Problem **1**(a):

The conjugacy relations $\hat{A}^\dagger_{\mathbf{k},\lambda} = -\hat{A}_{-\mathbf{k},\lambda}$, $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$ follow from hermiticity of the $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$ quantum fields and from the third eq. (6) for the polarization vectors:

$$\hat{A}^\dagger_{\mathbf{k},\lambda} = \int d^3\mathbf{x}\, e^{+i\mathbf{k}\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k}) \cdot \hat{\mathbf{A}}^\dagger(\mathbf{x}) = \int d^3\mathbf{x}\, e^{-i(-\mathbf{k})\mathbf{x}} (-\mathbf{e}^*_\lambda(-\mathbf{k})) \cdot \hat{\mathbf{A}} = -\hat{A}_{-\mathbf{k},\lambda}\,, \tag{S.1}$$

and likewise

$$\hat{E}^\dagger_{\mathbf{k},\lambda} = \int d^3\mathbf{x}\, e^{+i\mathbf{k}\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k}) \cdot \hat{\mathbf{E}}^\dagger(\mathbf{x}) = \int d^3\mathbf{x}\, e^{-i(-\mathbf{k})\mathbf{x}} (-\mathbf{e}^*_\lambda(-\mathbf{k})) \cdot \hat{\mathbf{E}} = -\hat{E}_{-\mathbf{k},\lambda}\,, \tag{S.2}$$

The equal-time commutation relations follow from eqs. (1): Obviously,

$$[\hat{A}_{\mathbf{k},\lambda}, \hat{A}_{\mathbf{k}',\lambda'}] = 0, \qquad [\hat{E}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] = 0. \tag{S.3}$$

Less obviously,

$$\begin{aligned}
[\hat{A}_{\mathbf{k},\lambda}, \hat{E}^\dagger_{\mathbf{k}',\lambda'}] &= \int d^3\mathbf{x} \int d^3\mathbf{y}\, e^{-i\mathbf{k}\mathbf{x}} \big(\mathbf{e}^*_\lambda(\mathbf{k})\big)^i \times e^{+i\mathbf{k}'\mathbf{y}} \big(\mathbf{e}_{\lambda'}(\mathbf{k}')\big)^j \times [\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] \\
&= \int d^3\mathbf{x} \int d^3\mathbf{y}\, e^{-i\mathbf{k}\mathbf{x}} \big(\mathbf{e}^*_\lambda(\mathbf{k})\big)^i \times e^{+i\mathbf{k}'\mathbf{y}} \big(\mathbf{e}_{\lambda'}(\mathbf{k}')\big)^j \times (-i)\delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ij} \\
&= -i \int d^3\mathbf{x}\, e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{x}} \times \big(\mathbf{e}^*_\lambda(\mathbf{k}) \cdot \mathbf{e}_{\lambda'}(\mathbf{k}')\big) \\
&= -i(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \times \big(\mathbf{e}^*_\lambda(\mathbf{k}) \cdot \mathbf{e}_{\lambda'}(\mathbf{k})\big) \\
&= -i(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \times \delta_{\lambda,\lambda'}\,,
\end{aligned} \tag{S.4}$$

or equivalently,

$$[\hat{A}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] = +i(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}') \times \delta_{\lambda,\lambda'}\,. \tag{S.5}$$

Problem **1**(b):

There are four terms in the Hamiltonian density (2), so let us consider them one by one. Combining Fourier transform with decomposition into polarization modes it is easy to see that in light of eq. (4),

$$\int d^3\mathbf{x}\,\hat{\mathbf{E}}^2(\mathbf{x}) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \tag{S.6}$$

and likewise

$$\int d^3\mathbf{x}\,\hat{\mathbf{A}}^2(\mathbf{x}) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\,. \tag{S.7}$$

Furthermore, using eq. (5) we have

$$\nabla\times\hat{\mathbf{A}}(\mathbf{x}) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{kx}}\Big(i\mathbf{k}\times\mathbf{e}_\lambda(\mathbf{k}) = \lambda|\mathbf{k}|\mathbf{e}_\lambda(\mathbf{k})\Big)\,\hat{A}_{\mathbf{k},\lambda} \tag{S.8}$$

and hence

$$\int d^3\mathbf{x}\,\Big(\nabla\times\hat{\mathbf{A}}(\mathbf{x})\Big)^2 \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \lambda^2\mathbf{k}^2\,\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\,. \tag{S.9}$$

Finally, the first eq. (6) gives us

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{x}) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{kx}}\Big(i\mathbf{k}\cdot\mathbf{e}_\lambda(\mathbf{k}) = i|\mathbf{k}|\delta_{\lambda,0}\Big)\,\hat{E}_{\mathbf{k},\lambda} \tag{S.10}$$

and hence

$$\int d^3\mathbf{x}\,\Big(\nabla\cdot\hat{\mathbf{E}}(\mathbf{x})\Big)^2 \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathbf{k}^2\hat{E}^\dagger_{\mathbf{k},0}\hat{E}_{\mathbf{k},0}\,. \tag{S.11}$$

In light of all these formulæ, we assemble the Hamiltonian (2) as

$$\hat{H} \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\left(\left(\frac{1}{2}+\frac{\mathbf{k}^2}{2m^2}\,\delta_{\lambda,0} = \frac{C_{\mathbf{k},\lambda}}{2}\right)\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \;+\; \left(\frac{m^2+\lambda^2\mathbf{k}^2}{2} \;=\; \frac{\omega^2_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}}\right)\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\right). \tag{8}$$

Problem **1**(c):

Given eqs. (S.3) and (S.5), we have

$$\begin{aligned}
[\hat a_{\mathbf{k},\lambda},\hat a_{\mathbf{k}',\lambda'}] \;&=\; -i\omega_{\mathbf{k}}\sqrt{\frac{C_{\mathbf{k}',\lambda'}}{C_{\mathbf{k},\lambda}}}\,\Big([\hat A_{\mathbf{k},\lambda},\hat E_{\mathbf{k}',\lambda'}]=(+i)(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\delta_{\lambda,\lambda'}\Big)\\
&\quad+\; -i\omega_{\mathbf{k}'}\sqrt{\frac{C_{\mathbf{k},\lambda}}{C_{\mathbf{k}',\lambda'}}}\,\Big([\hat E_{\mathbf{k},\lambda},\hat A_{\mathbf{k}',\lambda'}]=(-i)(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\delta_{\lambda,\lambda'}\Big)\\
&=\;\omega_{\mathbf{k}}\times(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\delta_{\lambda,\lambda'}\;-\;\omega_{\mathbf{k}'}\times(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\delta_{\lambda,\lambda'}\;=\;0
\end{aligned}\tag{S.12}$$

and likewise $[\hat a^\dagger_{\mathbf{k},\lambda},\hat a^\dagger_{\mathbf{k}',\lambda'}]=0$. On the other hand,

$$\begin{aligned}
[\hat a_{\mathbf{k},\lambda},\hat a^\dagger_{\mathbf{k}',\lambda'}] \;&=\; +i\omega_{\mathbf{k}}\sqrt{\frac{C_{\mathbf{k}',\lambda'}}{C_{\mathbf{k},\lambda}}}\,\Big([\hat A_{\mathbf{k},\lambda},\hat E^\dagger_{\mathbf{k}',\lambda'}]=(-i)(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\Big)\\
&\quad+\; -i\omega_{\mathbf{k}'}\sqrt{\frac{C_{\mathbf{k},\lambda}}{C_{\mathbf{k}',\lambda'}}}\,\Big([\hat E_{\mathbf{k},\lambda},\hat A^\dagger_{\mathbf{k}',\lambda'}]=(+i)(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\Big)\\
&=\;\omega_{\mathbf{k}}\times(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\;+\;\omega_{bk'}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\\
&=\;2\omega_{\mathbf{k}}\times(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\,.
\end{aligned}\tag{S.13}$$

$\mathcal{Q.E.D.}$

Problem **1**(d):

Expanding the operators $\hat a_{\mathbf{k},\lambda}$ and $\hat a^\dagger_{\mathbf{k},\lambda}$ according to eqs. (9), we have

$$\hat a^\dagger_{\mathbf{k},\lambda}\hat a_{\mathbf{k},\lambda}\;=\;\frac{\omega^2_{\mathbf{k}}}{C_{\mathbf{k},\lambda}}\,\hat A^\dagger_{\mathbf{k},\lambda}\hat A_{\mathbf{k},\lambda}\;+\;C_{\mathbf{k},\lambda}\,\hat E^\dagger_{\mathbf{k},\lambda}\hat E_{\mathbf{k},\lambda}\;+\;i\omega_{\mathbf{k}}\,\hat E^\dagger_{\mathbf{k},\lambda}\hat A_{\mathbf{k},\lambda}\;-\;i\omega_{\mathbf{k}}\,\hat A^\dagger_{\mathbf{k},\lambda}\hat E_{\mathbf{k},\lambda}\tag{S.14}$$

and therefore

$$\hat H^{\text{eq. (8)}}\;=\int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda\omega_{\mathbf{k}}\hat a^\dagger_{\mathbf{k},\lambda}\hat a_{\mathbf{k},\lambda}\;+\;\Delta\hat H\tag{S.15}$$

where

$$\Delta\hat H\;=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{i\omega_{\mathbf{k}}}{2}\sum_\lambda\Big(\hat A^\dagger_{\mathbf{k},\lambda}\hat E_{\mathbf{k},\lambda}\;-\;\hat E^\dagger_{\mathbf{k},\lambda}\hat A_{\mathbf{k},\lambda}\Big)\tag{S.16}$$

Thus, to prove eq. (10) we need to show that $\Delta\hat H$ is a c-number constant.

The trick here is to change the integration variable $\mathbf{k} \to -\mathbf{k}$ in the the first term in the integrand of eq. (S.16) and then apply eqs. (S.1) and (S.2):

$$\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \frac{i\omega_\mathbf{k}}{2}\,\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \frac{i\omega_\mathbf{k}}{2}\,\hat{A}^\dagger_{-\mathbf{k},\lambda}\hat{E}_{-\mathbf{k},\lambda} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \frac{i\omega_\mathbf{k}}{2}\,\hat{A}_{+\mathbf{k},\lambda}\hat{E}^\dagger_{+\mathbf{k},\lambda}\,. \tag{S.17}$$

Consequently

$$\begin{aligned}\Delta\hat{H} &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \frac{i\omega_\mathbf{k}}{2}\Big(\hat{A}_{\mathbf{k},\lambda}\hat{E}^\dagger_{\mathbf{k},\lambda} \,-\, \hat{E}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda} \;\equiv\; [\hat{A}_{\mathbf{k},\lambda},\hat{E}^\dagger_{\mathbf{k},\lambda}]\Big)\\ &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \frac{\omega_\mathbf{k}}{2}\,(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}=\mathbf{0})\\ &\equiv\; E_{\text{vacuum}}\end{aligned} \tag{S.18}$$

which is indeed a c-number constant, albeit divergent. $\mathcal{Q.E.D.}$

Physically, the vacuum energy (S.18) is the net zero-point energy of all the oscillatory modes of the vector field theory. This energy is infinite for two reasons, one having do do with the infinite volume of space and the other with its perfect continuity. The infinite-volume divergence of $\int d^3\mathbf{x}$ of a constant vacuum energy *density* manifest itself via the $(2\pi)^3\delta^{(3)}(\mathbf{0})$ factor, which is simply the Fourier transform of $\int d^3\mathbf{x}(1)$. Indeed, had we quantized the theory in a very large but finite box, we would have obtained the L^3 volume factor in eq. (S.18) instead of the delta function. In other words, the vacuum has energy density

$$\frac{\text{Energy}}{\text{Volume}}\bigg|_{\text{Vacuum}} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,3\times\frac{\omega_\mathbf{k}}{2}\,. \tag{S.19}$$

Alas, this integral diverges at large momenta so the vacuum energy *density* is also infinite. This is a generic problem of all Quantum Field Theories in a perfectly continuous space (and hence unlimitedly high momenta). Ultimately, this problem should be resolved by the fundamental theory of physics at ultra-short distances, whatever such theory might be.

Fortunately, for all practical purposes, we may safely disregard any c-number constant term in the Hamiltonian, even if such term is infinite — and that is exactly what we shall do in this course!

Problem $\mathbf{1}$(e):

Reversing eqs. (9), we have

$$\hat{A}_{\mathbf{k},\lambda} \;=\; \frac{\sqrt{C_{\mathbf{k},\lambda}}}{2\omega_{\mathbf{k}}}\left(\hat{a}_{\mathbf{k},\lambda} - \hat{a}^\dagger_{-\mathbf{k},\lambda}\right) \tag{S.20}$$

and therefore

$$\begin{aligned}\hat{\mathbf{A}}(\mathbf{x}) \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{e^{i\mathbf{k}\mathbf{x}}}{2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\,\mathbf{e}_\lambda(\mathbf{k})\left(\hat{a}_{\mathbf{k},\lambda} - \hat{a}^\dagger_{-\mathbf{k},\lambda}\right)\\ &=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{e^{+i\mathbf{k}\mathbf{x}}}{2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\,\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda} \;-\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{e^{-i\mathbf{k}\mathbf{x}}}{2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\,\mathbf{e}_\lambda(-\mathbf{k})\,\hat{a}_{+\mathbf{k},\lambda}\\ &=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\left(e^{+i\mathbf{k}\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda} \;+\; e^{-i\mathbf{k}\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}\right).\end{aligned} \tag{S.21}$$

It remains to work out the time dependence in the Heisenberg picture. For the free field governed by Hamiltonian (10), $\hat{a}_{\mathbf{k},\lambda}(t) = e^{-i\omega t}\hat{a}_{\mathbf{k},\lambda}(0)$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}(t) = e^{+i\omega t}\hat{a}^\dagger_{\mathbf{k},\lambda}(0)$ where $\omega \equiv \omega_{\mathbf{k}}$. Substituting this time dependence into eq. (S.21) and switching to relativistic notations, we immediately arrive at

$$\hat{\mathbf{A}}(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\left(e^{-ikx}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_{\mathbf{k}}}. \tag{11}$$

$\mathcal{Q.E.D.}$

Problem $\mathbf{1}$(f):

The 3–scalar field $\hat{A}^0(x)$ is governed by eqs. (3) and (S.10),

$$\hat{A}^0(\mathbf{x},t) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{-i|\mathbf{k}|}{m^2}\,e^{i\mathbf{k}\mathbf{x}}\,\hat{E}_{\mathbf{k},0}(t). \tag{S.22}$$

Reversing eqs. (9) for the $\hat{E}_{\mathbf{k},\lambda}$ operator, we have

$$\hat{E}_{\mathbf{k},\lambda} \;=\; \frac{i/2}{\sqrt{C_{\mathbf{k},\lambda}}}\left(\hat{a}_{\mathbf{k},\lambda} + \hat{a}^\dagger_{-\mathbf{k},\lambda}\right), \tag{S.23}$$

and in particular

$$\hat{E}_{\mathbf{k},0} \;=\; \frac{im}{2\omega_{\mathbf{k}}}\left(\hat{a}_{\mathbf{k},0}+\hat{a}^\dagger_{-\mathbf{k},0}\right).$$

Hence,

$$\begin{aligned}
\hat{A}^0(\mathbf{x}) &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\,\frac{|\mathbf{k}|}{m}\,e^{+i\mathbf{kx}}\left(\hat{a}_{\mathbf{k},0}+\hat{a}^\dagger_{-\mathbf{k},0}\right)\\
&= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\,\frac{|\mathbf{k}|}{m}\,e^{+i\mathbf{kx}}\,\hat{a}_{\mathbf{k},0} \;+\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\,\frac{|\mathbf{k}|}{m}\,e^{-i\mathbf{kx}}\,\hat{a}^\dagger_{+\mathbf{k},0} \qquad\qquad (S.24)\\
&= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\,\frac{|\mathbf{k}|}{m}\left(e^{+i\mathbf{kx}}\,\hat{a}_{\mathbf{k},0} \;+\; e^{-i\mathbf{kx}}\,\hat{a}^\dagger_{\mathbf{k},0}\right).
\end{aligned}$$

As to the time dependence, it works exactly as in eq. (11) for the vector field, thus

$$\hat{A}^0(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\,\frac{|\mathbf{k}|}{m}\left(e^{-ikx}\,\hat{a}_{\mathbf{k},0}(0) \;+\; e^{+ikx}\,\hat{a}^\dagger_{\mathbf{k},0}(0)\right)_{k^0=+\omega_{\mathbf{k}}}. \qquad (S.25)$$

In light of similarity between eqs. (11) and (S.25), we may combine them into a single eq. (12) where

$$\mathbf{f}(\mathbf{k},\lambda) \;=\; C_{\mathbf{k},\lambda}\,\mathbf{e}_\lambda(\mathbf{k}) \quad\text{and}\quad f^0(\mathbf{k},\lambda) \;=\; \frac{|\mathbf{k}|}{m}\,\delta_{\lambda,0} \qquad (S.26)$$

or equivalently, (13).

<u>Problem **1**(g):</u>

According to eq. (12),

$$\begin{aligned}
(\partial^2+m^2)\hat{A}^\mu(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda\Big(&(-k^2+m^2)e^{-ikx}f^\mu(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda}(0) \qquad (S.27)\\
&+\;(-k^2+m^2)e^{+ikx}f^{*\mu}(\mathbf{k},\lambda)\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\Big)_{k^0=+\omega_{\mathbf{k}}},
\end{aligned}$$

which vanishes because $(-k^2+m^2)=0$ for $k^0=\omega_{\mathbf{k}}$. Likewise,

$$\begin{aligned}
\partial_\mu\hat{A}^\mu(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda\Big(&e^{-ikx}\big(ik_\mu f^\mu(\mathbf{k},\lambda)\big)\hat{a}_{\mathbf{k},\lambda}(0) \qquad (S.28)\\
&+\;e^{+ikx}\big(-ik_\mu f{*}^\mu(\mathbf{k},\lambda)\big)\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\Big)_{k^0=+\omega_{\mathbf{k}}},
\end{aligned}$$

which vanishes because $k_\mu f^\mu(\mathbf{k},\lambda)=0$ for all polarizations λ. $\mathcal{Q.E.D.}$

Problem **2**(a):

The simplest way to prove this lemma is by direct inspection, component by component:

$$\begin{aligned}
\sum_\lambda f^i(\mathbf{k},\lambda)f^{*j}(\mathbf{k},\lambda) &= \sum_\lambda e^i_\lambda(\mathbf{k})e^{*j}_\lambda(\mathbf{k}) \;+\; \frac{\mathbf{k}^2}{m^2}\,e^i_0(\mathbf{k})e^{*j}_0(\mathbf{k}) \;=\; \delta^{ij} \;+\; \frac{k^ik^j}{m^2}\,;\\
\sum_\lambda f^i(\mathbf{k},\lambda)f^{*0}(\mathbf{k},\lambda) &= f^i(\mathbf{k},0)f^{*0}(\mathbf{k},0) \;=\; \frac{k^i\omega_{\mathbf{k}}}{m^2}\,;\\
\sum_\lambda f^0(\mathbf{k},\lambda)f^{*0}(\mathbf{k},\lambda) &= \left|f^0(\mathbf{k},0)\right|^0 \;=\; \frac{\mathbf{k}^2}{m^2} \;=\; -1 \;+\; \frac{\omega^2_{\mathbf{k}}}{m^2}\,.
\end{aligned} \tag{S.29}$$

Alternatively, we may use the fact that the three four-vectors $f^\mu(\mathbf{k},\lambda)$ (fixed $\mathbf{k}$, $\lambda=-1,0,+1$) are orthogonal to each other and also to the $k^\mu=(\omega_{\mathbf{k}},\mathbf{k})$. Furthermore, each $\big(f(\mathbf{k},\lambda)\big)^2=-1$. Consequently, the symmetric matrix (in Lorentz indices μ,ν) on the left hand side of eq. (14) has to be (minus) the projection matrix onto four-vectors orthogonal to the k^μ, and that is precisely the matrix appearing on the right hand side of eq. (14) (note $k^2=m^2$).

Problem **2**(b):

The operator product $\hat{A}^\mu(x)\hat{A}^\nu(y)$ comprises $\hat{a}\hat{a}$, $\hat{a}^\dagger\hat{a}^\dagger$, $\hat{a}^\dagger\hat{a}$ and $\hat{a}\hat{a}^\dagger$ terms. The first three kinds of terms have zero matrix elements between vacuum states while $\langle 0|\,\hat{a}_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k}',\lambda'}\,|0\rangle = 2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}$. Consequently,

$$\begin{aligned}
\langle 0|\,\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\sum_\lambda\left[e^{-ik(x-y)}\,f^\mu(\mathbf{k},\lambda)f^{*\nu}(\mathbf{k},\lambda)\right]_{k^0=+\omega_{\mathbf{k}}}\\
&= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left[\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)e^{-ik(x-y)}\right]_{k^0=+\omega_{\mathbf{k}}}\\
&= \left(-g^{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right)\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left[e^{-ik(x-y)}\right]_{k^0=+\omega_{\mathbf{k}}}\\
&\equiv \left(-g^{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right)D(x-y).
\end{aligned} \tag{15}$$

Problem $\mathbf{2}$(c):

Starting with eq. (15), we immediately see that for the un-modified time-ordering,

$$\begin{aligned}\langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;&=\; \theta(x^0-y^0)\,\langle 0|\,\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;+\; \theta(y^0-x^0)\,\langle 0|\,\hat{A}^\nu(y)\hat{A}^\mu(x)\,|0\rangle \\ &=\; \theta(x^0-y^0)\left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)D(x-y) \\ &\quad+\; \theta(y^0-x^0)\left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)D(y-x).\end{aligned} \tag{S.30}$$

On the other hand,

$$G_F(x-y) \;=\; \theta(x^0-y^0)D(x-y) \;+\; \theta(y^0-x^0)D(y-x), \tag{S.31}$$

and hence

$$\begin{aligned}\left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)G_F(x-y) \;&=\; \theta(x^0-y^0)\left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)D(x-y) \\ &\quad+\; \theta(y^0-x^0)\left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)D(y-x) \\ &\quad-\; i\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y)\end{aligned} \tag{S.32}$$

where the δ-function term arises from taking time derivatives of the θ-functions. (*cf.* explanation of $(\partial^2+m^2)G_F(x-y)=-i\delta^{(4)}(x-y)$ in class.) Comparing eqs. (S.30) and (S.32), we obtain

$$\langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;=\; \left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)G_F(x-y) \;-\; i\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y) \tag{S.33}$$

and hence

$$\langle 0|\,\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;=\; \left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)G_F(x-y). \tag{S.34}$$

This proves the first line of eq. (16); the second line follows from the momentum-space form of the scalar propagator

$$\left(-g^{\mu\nu} - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)\left[G_F(x-y) = \int\frac{d^4\mathbf{k}}{(2\pi)^4}\,\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\right]$$
$$= \int\frac{d^4\mathbf{k}}{(2\pi)^4}\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\,. \tag{S.35}$$

$\mathcal{Q.E.D.}$