

Problem 3(a):

First, let us calculate the commutator $[\gamma^\kappa \gamma^\lambda, \gamma^\nu]$. According to eq. (18), any product of 2 Dirac matrices can be re-ordered as

$$\gamma^\nu \gamma^\mu = 2g^{\mu\nu} - \gamma^\mu \gamma^\nu. \quad (\text{S.1})$$

Applying this formula twice, we have

$$\gamma^\kappa \gamma^\lambda \gamma^\mu = \gamma^\kappa (2g^{\lambda\mu} - \gamma^\mu \gamma^\lambda) = 2g^{\lambda\mu} \gamma^\kappa - (\gamma^\kappa \gamma^\mu) \gamma^\lambda = 2g^{\lambda\mu} \gamma^\kappa - 2g^{\kappa\mu} \gamma^\lambda + \gamma^\mu \gamma^\kappa \gamma^\lambda \quad (\text{S.2})$$

and hence

$$[\gamma^\kappa \gamma^\lambda, \gamma^\mu] = 2g^{\lambda\mu} \gamma^\kappa - 2g^{\kappa\mu} \gamma^\lambda. \quad (\text{S.3})$$

Next, according to eq. (S.1), $[\gamma^\mu, \gamma^\nu] = 2\gamma^\mu \gamma^\nu - 2g^{\mu\nu}$ and hence

$$S^{\mu\nu} = -\frac{i}{2}(\gamma^\mu \gamma^\nu - g^{\mu\nu}). \quad (\text{S.4})$$

Consequently,

$$[S^{\kappa\lambda}, \gamma^\mu] = -\frac{i}{2}[\gamma^\kappa \gamma^\lambda, \gamma^\mu] = -ig^{\lambda\mu} \gamma^\kappa + ig^{\kappa\mu} \gamma^\lambda \quad (\text{20})$$

or equivalently

$$[\gamma^\mu, S^{\kappa\lambda}] = ig^{\lambda\mu} \gamma^\kappa - ig^{\kappa\mu} \gamma^\lambda. \quad (\text{S.5})$$

Finally, by Leibniz rule,

$$\begin{aligned} [\gamma^\kappa \gamma^\lambda, S^{\mu\nu}] &= \gamma^\kappa [\gamma^\lambda, S^{\mu\nu}] + [\gamma^\kappa, S^{\mu\nu}] \gamma^\lambda \\ &= \gamma^\kappa (ig^{\lambda\mu} \gamma^\nu - ig^{\lambda\nu} \gamma^\mu) + (ig^{\kappa\mu} \gamma^\nu - ig^{\kappa\nu} \gamma^\mu) \gamma^\lambda \\ &= ig^{\lambda\mu} \gamma^\kappa \gamma^\nu - ig^{\kappa\nu} \gamma^\mu \gamma^\lambda - ig^{\lambda\nu} \gamma^\kappa \gamma^\mu + ig^{\kappa\mu} \gamma^\nu \gamma^\lambda \\ &= ig^{\lambda\mu} (\gamma^\kappa \gamma^\nu - g^{\kappa\nu}) - ig^{\kappa\nu} (\gamma^\mu \gamma^\lambda - g^{\lambda\mu}) \\ &\quad - ig^{\lambda\nu} (\gamma^\kappa \gamma^\mu - g^{\kappa\mu}) + ig^{\kappa\mu} (\gamma^\nu \gamma^\lambda - g^{\lambda\nu}) \\ &= 2g^{\lambda\mu} S^{\kappa\nu} - 2g^{\kappa\nu} S^{\mu\lambda} - 2g^{\lambda\nu} S^{\kappa\mu} + 2g^{\kappa\mu} S^{\nu\lambda}, \end{aligned} \quad (\text{S.6})$$

and therefore,

$$\begin{aligned}
[S^{\kappa\lambda}, S^{\mu\nu}] &= \frac{i}{2} [\gamma^\kappa \gamma^\lambda, S^{\mu\nu}] \\
&= ig^{\lambda\mu} S^{\kappa\nu} - ig^{\kappa\nu} S^{\mu\lambda} - ig^{\lambda\nu} S^{\kappa\mu} + ig^{\kappa\mu} S^{\nu\lambda} \\
&= ig^{\lambda\mu} S^{\kappa\nu} + ig^{\kappa\nu} S^{\lambda\mu} - ig^{\lambda\nu} S^{\kappa\mu} - ig^{\kappa\mu} S^{\lambda\nu}.
\end{aligned} \tag{21}$$

Q.E.D.

Problem 3(b):

$$\begin{aligned}
\{\gamma^\rho, \gamma^\lambda \gamma^\mu \gamma^\nu\} &= 2g^{\rho\lambda} \gamma^\mu \gamma^\nu - 2g^{\rho\mu} \gamma^\lambda \gamma^\nu + 2g^{\rho\nu} \gamma^\lambda \gamma^\mu, \\
[\gamma^\rho, \gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu] &= 2g^{\rho\kappa} \gamma^\lambda \gamma^\mu \gamma^\nu - 2g^{\rho\lambda} \gamma^\kappa \gamma^\mu \gamma^\nu + 2g^{\rho\mu} \gamma^\kappa \gamma^\lambda \gamma^\nu - 2g^{\rho\nu} \gamma^\kappa \gamma^\lambda \gamma^\mu, \\
[S^{\rho\sigma}, \gamma^\lambda \gamma^\mu \gamma^\nu] &= ig^{\sigma\lambda} \gamma^\rho \gamma^\mu \gamma^\nu + ig^{\sigma\mu} \gamma^\lambda \gamma^\rho \gamma^\nu + ig^{\sigma\nu} \gamma^\lambda \gamma^\mu \gamma^\rho \\
&\quad - ig^{\rho\lambda} \gamma^\sigma \gamma^\mu \gamma^\nu - ig^{\rho\mu} \gamma^\lambda \gamma^\sigma \gamma^\nu - ig^{\rho\nu} \gamma^\lambda \gamma^\mu \gamma^\sigma.
\end{aligned}$$

The algebra is straightforward.

Problem 3(c):

$$\begin{aligned}
\gamma^\alpha \gamma_\alpha &= \frac{1}{2} \{\gamma^\alpha, \gamma^\beta\} g_{\alpha\beta} = g^{\alpha\beta} g_{\alpha\beta} = 4; \\
\gamma^\alpha \gamma^\nu \gamma_\alpha &= 2g^{\alpha\nu} \gamma_\alpha - \gamma^\nu \gamma^\alpha \gamma_\alpha = 2\gamma^\nu - \gamma^\nu(4) = -2\gamma^\nu; \\
\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha &= 2g^{\alpha\mu} \gamma^\nu \gamma_\alpha - \gamma^\mu \gamma^\alpha \gamma^\nu \gamma_\alpha = 2\gamma^\nu \gamma^\mu - \gamma^\mu(-2\gamma^\nu) = 2\{\gamma^\nu, \gamma^\mu\} = 4g^{\mu\nu}; \\
\gamma^\alpha \gamma^\lambda \gamma^\mu \gamma^\nu \gamma_\alpha &= 2g^{\alpha\lambda} \gamma^\mu \gamma^\nu \gamma_\alpha - \gamma^\lambda \gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 2\gamma^\mu \gamma^\nu \gamma^\lambda - \gamma^\lambda(4g^{\mu\nu}) \\
&= 2(\gamma^\mu \gamma^\nu - 2g^{\mu\nu}) \gamma^\lambda = -2\gamma^\nu \gamma^\mu \gamma^\lambda.
\end{aligned} \tag{S.7}$$

Problem 3(d): Let $F = -\frac{i}{2} \Theta_{\alpha\beta} S^{\alpha\beta}$, thus $M = e^F$ and $M^{-1} = e^{-F}$. We shall use the multiple-commutator formula for the $e^{-F} \gamma^\mu e^{+F}$, so we begin by evaluating the single commutator

$$[\gamma^\mu, F] = -\frac{i}{2} \Theta_{\alpha\beta} [\gamma^\mu, S^{\alpha\beta}] = \frac{1}{2} \Theta_{\alpha\beta} (g^{\mu\alpha} \gamma^\beta - g^{\mu\beta} \gamma^\alpha) = \Theta_{\alpha\beta} g^{\mu\alpha} \gamma^\beta = \Theta^\mu_\beta \gamma^\beta. \tag{S.8}$$

The multiple commutators follow immediately from this formula,

$$\begin{aligned}
[[\gamma^\mu, F], F] &= \Theta^\mu_\lambda \Theta^\lambda_\nu \gamma^\nu, \\
[[[\gamma^\mu, F], F], F] &= \Theta^\mu_\lambda \Theta^\lambda_\rho \Theta^\rho_\nu \gamma^\nu, \\
&\dots\dots\dots
\end{aligned}
\tag{S.9}$$

Therefore, by the multiple commutator formula,

$$\begin{aligned}
M^{-1} \gamma^\mu M &= e^{-F} \gamma^\mu e^{+F} \\
&= \gamma^\mu + [\gamma^\mu, F] + \frac{1}{2} [[\gamma^\mu, F], F] + \frac{1}{6} [[[\gamma^\mu, F], F], F] + \dots \\
&= \gamma^\mu + \Theta^\mu_\nu \gamma^\nu + \frac{1}{2} \Theta^\mu_\lambda \Theta^\lambda_\nu \gamma^\nu + \frac{1}{6} \Theta^\mu_\lambda \Theta^\lambda_\rho \Theta^\rho_\nu \gamma^\nu + \dots \\
&= L^\mu_\nu \gamma^\nu.
\end{aligned}
\tag{S.10}$$

Q.E.D.