Problem **1**(a):

In 3–vector notations, the Lorentz algebra comprises the $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ operators satisfying the commutation relations

$$\big[\hat{J}^i,\hat{J}^j\big] \;=\; i\epsilon^{ij\ell}\,\hat{J}^\ell,\quad \big[\hat{J}^i,\hat{K}^j\big] \;=\; i\epsilon^{ij\ell}\,\hat{K}^\ell,\quad \big[\hat{K}^i,\hat{K}^j\big] \;=\; -i\epsilon^{ij\ell}\,\hat{J}^\ell. \tag{S.1}$$

Consequently, for the $\hat{\mathbf{J}}_\pm = \frac{1}{2}\big(\hat{\mathbf{J}}\pm i\hat{\mathbf{K}}\big)$, we have

$$\big[\hat{J}^i_\pm,\hat{J}^j_\pm\big] \;=\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;\mp\; \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \;\mp\; \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \;+\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;=\; i\epsilon^{ij\ell}\,\hat{J}^\ell_\pm \tag{S.2}$$

while

$$\big[\hat{J}^i_\pm,\hat{J}^j_\mp\big] \;=\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;\mp\; \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \;\pm\; \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \;-\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;=\; 0. \tag{S.3}$$

$\mathcal{Q.E.D.}$

Problem **1**(b):

First, note the hermiticity of the σ^μ matrices and the fact that any hermitian 2×2 matrix is a unique linear combination of the four σ_ν with real coefficients. Consequently,

$$\forall M:\; M\sigma^\mu M^\dagger \;=\; \sigma^\nu\,L_\nu^{\;\mu}(M) \Longrightarrow\; X'_\nu \;=\; L_\nu^{\;\mu}(M)X_\mu \tag{S.4}$$

for some real 4×4 matrix $L_\nu^{\;\mu}(M)$. Furthermore, for $M\in SL(2,\mathbf{C})$, *i.e.* for $\det(M)=1$, this $L_\nu^{\;\mu}(M)$ matrix defines a Lorentz transform for which $X'_\mu X'^\mu = X_\mu X^\mu$. To see this, we note that

$$\det\big(X_\mu\sigma^\mu\big) \;=\; \det\begin{pmatrix} X_0+X_3 & X_1-iX_2\\ X_1+iX_2 & X_0-X_3\end{pmatrix} \;=\; (X_0)^2-(X_3)^2-(X_1)^2-(X_2)^2 \;\equiv X^2 \tag{S.5}$$

and then calculate

$$X'^2 \;=\; \det\big(X'_\mu\sigma^\mu\big) \;=\; \det\big(M(X_\mu\sigma^\mu)M^\dagger\big) \;=\; |\det(M)|^2\times\det\big(X_\mu\sigma^\mu\big) \;=\; 1\times X^2. \tag{S.6}$$

Also, the Lorentz transform $X_\mu\to X'_\mu = L_\mu^{\;\nu}X_\nu$ is orthochronous because

$$L_0^{\;0} \;=\; \tfrac{1}{2}\,\mathrm{tr}\big(\sigma^\nu L_\nu^{\;0}\big) \;=\; \tfrac{1}{2}\,\mathrm{tr}\big(M\sigma^0 M^\dagger\big) \;=\; \tfrac{1}{2}\,\mathrm{tr}\big(MM^\dagger\big) \;>\; 0. \tag{S.7}$$

Problem $\mathbf{1}$(b*):

The simplest proof the $L_\mu^{\ \nu}(M)$ is proper as well as orthochronous involves the group law (problem $\mathbf{2}$(c) below) and the explicit examples of a pure rotation and a pure boost (problem $\mathbf{2}$(d) below, eqs. (S.11) and (S.13)), both of which are manifestly proper.

For any $SL(2,\mathbf{C})$ matrix M we may decompose $M = HU$ where $H = \sqrt{MM^\dagger}$ is hermitian and $U = H^{-1}M$ is unitary. (Proof: $UU\dagger = H^{-1}MM^\dagger H^{-1} = H^{-1}H^2H^{-1} = 1$.) Furthermore, both H and U are unimodular ($\det(H) = \det(U) = 1$), or in other words $H, U \in SL(2,\mathbf{C})$, which allows us to define two separate Lorentz transforms $L(H)$ and $L(U)$. According to the group law, together these two transform accomplish the $L(M)$ transform,

$$L(M) \;=\; L(H)\times L(U). \tag{S.8}$$

Now, H is hermitian, unimodular, and positive definite, hence it has a well-defined logarithm which is hermitian and traceless, $\operatorname{tr}(\log H) = 0$. For the 2×2 matrices, this means $\log H = -\frac{1}{2}\mathbf{r}\boldsymbol{\sigma}$ for some real 3–vector $\mathbf{r}$, or equivalently $H = \exp\left(-\frac{1}{2}r\mathbf{n}\boldsymbol{\sigma}\right)$. As we shall see in eq. (S.13) below, this means that $L(H)$ is a pure Lorentz boost of rapidity r in the direction $\mathbf{n}$. This boost manifestly does not invert space or time, thus $L(U)$ is proper.

Likewise, U is unitary and unimodular, thus $U \in SU(2)$ and defines a pure rotation of space. Indeed, any $U \in SU(2)$ can be written as $U = \exp\left(-\frac{i}{2}\theta\mathbf{n}'\boldsymbol{\sigma}\right)$ for some angle θ and some axis $\mathbf{n}'$, and according to eq. (S.11) below $L(U)$ is indeed a pure space rotation by angle θ around axis $\mathbf{n}'$. Again, this rotation is proper — it does not invert space or time. Thus, $L(H)$ and $L(U)$ are both proper Lorentz transforms, hence their product $L(M)$ must also be proper. (Proof: $\det(L(M)) = \det(L(H))\times\det(L(U)) = +1$.) $\mathcal{Q.E.D.}$

And by the way, since any proper, orthochronous Lorentz transform $L \in SO^+(1,3)$ can be realized as $L(M)$ for some $M \in SL(2,\mathbf{C})$, it follows that any such transform is a product of a pure space rotation $L(H)$ followed by a pure Lorentz boost $L(U)$.

Problem $\mathbf{1}$(c):

$$\begin{aligned}\sigma_\lambda\, L^\lambda_{\ \mu}(M_2M_1) \;&=\; (M_2M_1)\sigma_\mu(M_2M_1)^\dagger \;=\; M_2\left(M_1\sigma_\mu M_1^\dagger \;=\; \sigma_\nu\, L^\nu_{\ \mu}(M_1)\right)M_2^\dagger \\ &=\; \left(M_2\,\sigma_\nu\,M_2^\dagger\right)\,L^\nu_{\ \mu}(M_1) \;=\; \sigma_\lambda\,L^\lambda_{\ \nu}(M_2)\,L^\nu_{\ \mu}(M_1)\end{aligned} \tag{S.9}$$

and hence $L^\lambda_{\ \mu}(M_2M_1) = L^\lambda_{\ \nu}(M_2)\,L^\nu_{\ \mu}(M_1)$, *i.e.* $L(M_2M_1) = L(M_2)\,L(M_1)$. $\mathcal{Q.E.D.}$

Problem $\mathbf{1}$(d):

Let $M = \exp\bigl(-\frac{i}{2}\theta\,\mathbf{n}\boldsymbol{\sigma}\bigr) = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\,\mathbf{n}\boldsymbol{\sigma}$ and hence $M^\dagger = M^{-1} = cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\,\mathbf{n}\boldsymbol{\sigma}$. Given $\sigma^0 = 1$ and unitarity of M, we have $M\sigma^0 M^\dagger = \sigma^0$, and hence according to eq. (7) $t' = t$ regardless of $\mathbf{x}$. In other words, $L(M)$ does not affect the time and is a purely spatial rotation. Specifically,

$$\begin{aligned}\boldsymbol{\sigma}\cdot\mathbf{x}' \;=\; M(\mathbf{x}\boldsymbol{\sigma})M^\dagger \;&=\; \cos^2\tfrac{\theta}{2}\,(\mathbf{x}\boldsymbol{\sigma}) \;-\; i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\,\Bigl([\mathbf{n}\boldsymbol{\sigma},\mathbf{x}\boldsymbol{\sigma}] \;=\; 2i(\mathbf{n}\times\mathbf{x})\cdot\boldsymbol{\sigma}\Bigr)\\ &\qquad +\; \sin^2\tfrac{\theta}{2}\,\Bigl((\mathbf{n}\boldsymbol{\sigma})(\mathbf{x}\boldsymbol{\sigma})(\mathbf{n}\boldsymbol{\sigma}) \;=\; 2(\mathbf{n}\mathbf{x})(\mathbf{n}\boldsymbol{\sigma}) \;-\; (\mathbf{x}\boldsymbol{\sigma})\Bigr)\\ &=\; \cos\theta\,(\mathbf{x}\boldsymbol{\sigma}) \;+\; \sin\theta((\mathbf{n}\times\mathbf{x})\boldsymbol{\sigma}) \;+\; (1-\cos\theta)(\mathbf{n}\mathbf{x})(\mathbf{n}\boldsymbol{\sigma}),\end{aligned} \tag{S.10}$$

thus

$$\mathbf{x}' \;=\; \cos\theta\bigl(\mathbf{x}-\mathbf{n}(\mathbf{n}\mathbf{x})\bigr) \;+\; \sin\theta\,\mathbf{n}\times\mathbf{x} \;+\; \mathbf{n}(\mathbf{n}\mathbf{x}) \tag{S.11}$$

which indeed describes a rotation through angle θ around axis $\mathbf{n}$.

Now consider $M = \exp\bigl(-\frac{r}{2}\,\mathbf{n}\boldsymbol{\sigma}\bigr) = \cosh\frac{r}{2} - \sinh\frac{r}{2}\,\mathbf{n}\boldsymbol{\sigma}$ and hence $M^\dagger = M$. In this case, we have

$$\begin{aligned}M\,\bigl(x^\mu\sigma_\mu \;\equiv\; t-\mathbf{x}\boldsymbol{\sigma}\bigr)\,M^\dagger \;&=\; \cosh^2\tfrac{r}{2}\,(t-\mathbf{x}\boldsymbol{\sigma})\\ &\qquad -\; \sinh\tfrac{r}{2}\cosh\tfrac{r}{2}\,\Bigl(\bigl\{\mathbf{n}\boldsymbol{\sigma},(t-\mathbf{x}\boldsymbol{\sigma})\bigr\} \;=\; 2t(\mathbf{n}\boldsymbol{\sigma}) \;-\; 2(\mathbf{n}\mathbf{x})\Bigr)\\ &\qquad +\; \sinh^2\tfrac{r}{2}\,\Bigl((\mathbf{n}\boldsymbol{\sigma})(t-\mathbf{x}\boldsymbol{\sigma})(\mathbf{n}\boldsymbol{\sigma}) \;=\; t \;-\; 2(\mathbf{n}\mathbf{x})(\mathbf{n}\boldsymbol{\sigma}) \;+\; (\mathbf{x}\boldsymbol{\sigma})\Bigr)\\ &=\; \bigl(\cosh r\,t \;+\; \sinh r\,\mathbf{n}\mathbf{x}\bigr) \;-\; (\boldsymbol{\sigma}\mathbf{n})\bigl(\sinh r\,t \;+\; \cosh r\,\mathbf{n}\mathbf{x}\bigr)\\ &\qquad -\; \boldsymbol{\sigma}\cdot\bigl(\mathbf{x}-\mathbf{n}(\mathbf{n}\mathbf{x})\bigr),\end{aligned} \tag{S.12}$$

and therefore,

$$t' \;=\; (\cosh r)\,t \;+\; (\sinh r)\,\mathbf{n}\mathbf{x}, \quad \mathbf{x}' \;=\; \mathbf{n}\bigl((\sinh r)\,t \;+\; (\cosh r)\,\mathbf{n}\mathbf{x}\bigr) \;+\; \bigl(\mathbf{x}-\mathbf{n}(\mathbf{n}\mathbf{x})\bigr), \tag{S.13}$$

which is precisely the Lorentz boost of rapidity r in the direction $\mathbf{n}$. (The rapidity r is related to the usual parameters of a Lorentz boost according to $\beta = \tanh r$, $\gamma = \cosh r$, $\gamma\beta = \sinh r$. For several boosts in the same directions, the rapidities add up, $r_{\text{tot}} = r_1 + r_2 + \cdots$.) $\mathcal{Q.E.D.}$

Problem $\mathbf{1}$ (e):

For any Lie algebra equivalent to an angular momentum or its analytic continuation, the product of two doublets comprises a triplet and a singlet, $\mathbf{2}\otimes\mathbf{2}=\mathbf{3}\oplus\mathbf{1}$, or in (j) notations, $(\frac{1}{2})\otimes(\frac{1}{2})=(1)\oplus(0)$. Furthermore, the triplet $\mathbf{3}=(1)$ is symmetric with respect to permutations of the two doublets while the singlet $\mathbf{1}=(0)$ is antisymmetric.

For two separate and independent types of angular momenta $\mathbf{J}_+$ and $\mathbf{J}_-$ we combine the j_+ quantum numbers independently of j_- and the j_- quantum numbers independently of j_+. Thus,

$$(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2}) \;=\; (1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0). \tag{S.14}$$

Furthermore, the symmetric part of this product should be either symmetric with respect to both the j_+ and the j_- indices or antisymmetric with respect to both indices, thus

$$\left[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})\right]_{\rm sym} \;=\; (1,1)\oplus(0,0). \tag{S.15}$$

Likewise, the antisymmetric part is either symmetric with respect to the j_+ but antisymmetric with respect to the j_- or the other way around, thus

$$\left[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})\right]_{\rm antisym} \;=\; (1,0)\oplus(0,1). \tag{S.16}$$

From the $SO(1.3)$ point of view, the $(\frac{1}{2},\frac{1}{2})$ multiplet is the Lorentz vector, hence the generic 2–index Lorentz tensor decomposes into irreducible multiplets according to eq. (S.14). Imposing symmetry conditions, we have eq. (S.15) for the *symmetric* 2–index tensor $T^{\mu\nu}=T^{\nu\mu}$ where the singlet $(0,0)$ corresponds to the trace T^μ_μ while the $(1,1)$ irreducible multiplet is the traceless symmetric tensor.

Likewise, the antisymmetric Lorentz tensor $F^{\mu\nu}=-F^{\nu\mu}$ decomposes according to eq. (S.16). Here, the irreducible components $(1,0)$ and $(0,1)$ are complex but conjugate to each other; individually, they describe antisymmetric tensors subject to complex duality conditions $\frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}F_{\mu\nu}=\pm iF^{\kappa\lambda}$, *i.e.* $\mathbf{E}=\pm i\mathbf{B}$.

Problem **1**(f):

Without the $\gamma_\mu\Psi^\mu=0$ constraint, the spin-vector Ψ^μ_a is the tensor product or the Dirac spinor and the Lorentz vector, thus

$$\left[(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\right]\otimes(\tfrac{1}{2},\tfrac{1}{2}) \;=\; (1,\tfrac{1}{2})\oplus(0,\tfrac{1}{2})\oplus(\tfrac{1}{2},1)\oplus(\tfrac{1}{2},0). \tag{S.17}$$

The constraint removes a Dirac spinor $\gamma_\mu\Psi^\mu \;\Rightarrow\; (\frac{1}{2},0)\oplus(0,\frac{1}{2})$, thus we are left with the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ part for the Rarita–Schwinger spin-vector.

Problem **2**(a):

In spacetime, any purely spatial rotation $R(t,\mathbf{x}) = (t, R\mathbf{x})$ commutes with the space reflection $P(t,\mathbf{x}) = (t,-\mathbf{x})$. Consequently, by the group law, the parity operator $\hat{\mathcal{P}}$ in the Fock space must commute with operators $\hat{\mathcal{D}}(R)$ representing the space rotations. And since the spatial rotations are generated by the angular momentum components $\hat{J}^i$, $\hat{\mathcal{D}}(\theta,\mathbf{n}) = \exp(-i\theta\mathbf{n}\hat{\mathbf{J}})$, the fact that all the $\hat{\mathcal{D}}(\theta,\mathbf{n})$ commute with the parity operator implies that $\hat{\mathbf{J}}\hat{\mathcal{P}} = \hat{\mathcal{P}}\hat{\mathbf{J}}$.

Next, consider a pure Lorentz boost $B(r,\mathbf{n})$ (S.13). Clearly, reflecting the space reverses the direction of the boost,

$$PB(r,\mathbf{n})P \;=\; B(r,-\mathbf{n}) \;=\; B^{-1}(r,\mathbf{n}), \tag{S.18}$$

hence the operators

$$\hat{\mathcal{D}}(r,\mathbf{n}) \;=\; \exp(-ir\mathbf{n}\hat{\mathbf{K}}) \tag{S.19}$$

representing the pure Lorentz boosts in the Fock space must have similar commutation relations with the parity operator:

$$\hat{\mathcal{P}}\hat{\mathcal{D}}(r,\mathbf{n})\hat{\mathcal{P}} \;=\; \hat{\mathcal{D}}(r,-\mathbf{n}). \tag{S.20}$$

Consequently, in terms of the boost generators $\hat{K}^i$

$$\hat{\mathcal{P}}\exp(-ir\mathbf{n}\hat{\mathbf{K}})\hat{\mathcal{P}} \;=\; \exp(+ir\mathbf{n}\hat{\mathbf{K}}) \implies \hat{\mathcal{P}}\hat{\mathbf{K}}\hat{\mathcal{P}} \;=\; -\hat{\mathbf{K}}. \tag{S.21}$$

And in terms of the $\hat{\mathbf{J}}_\pm$ operators (1), the fact that $\hat{\mathcal{P}}$ commutes with $\hat{\mathbf{J}}$ but anticommutes with $\hat{\mathbf{K}}$ means

$$\hat{\mathcal{P}}\hat{\mathbf{J}}_+\hat{\mathcal{P}} \;=\; \hat{\mathbf{J}}_- \quad\text{and}\quad \hat{\mathcal{P}}\hat{\mathbf{J}}_-\hat{\mathcal{P}} \;=\; \hat{\mathbf{J}}_+\,; \tag{S.22}$$

in other words, parity interchanges the $\hat{\mathbf{J}}_+$ and the $\hat{\mathbf{J}}_-$ operators.

Now consider a multiplet $\hat{\varphi}_a(x)$ of quantum fields. Saying that this multiplets has definite values of $j_+ = C$ and $j_- = D$ technically means that

$$\begin{aligned} \left[\hat{\mathbf{J}}_+^2, \hat{\varphi}_a(x)\right] \;&=\; C(C+1)\hat{\varphi}_a(x), \\ \left[\hat{\mathbf{J}}_-^2, \hat{\varphi}_a(x)\right] \;&=\; D(D+1)\hat{\varphi}_a(x). \end{aligned} \tag{S.23}$$

Applying the parity operator to the $\varphi_a(x)$ fields, we have

$$\hat{\varphi}'_a(t,-\mathbf{x}) \;=\; \hat{\mathcal{P}}\hat{\varphi}_a(t,\mathbf{x})\hat{\mathcal{P}}. \tag{S.24}$$

Consequently,

$$\begin{aligned}
\left[\hat{\mathbf{J}}_+^2,\hat{\varphi}'_a(t,-\mathbf{x})\right] &= \hat{\mathcal{P}}\left[\hat{\mathcal{P}}\hat{\mathbf{J}}_+^2\hat{\mathcal{P}},\hat{\mathcal{P}}\hat{\varphi}'_a(t,-\mathbf{x})\hat{\mathcal{P}}\right]\hat{\mathcal{P}} && \langle\!\langle \text{note } \hat{\mathcal{P}}^2=1\rangle\!\rangle\\
&= \hat{\mathcal{P}}\left[\hat{\mathbf{J}}_-^2,\hat{\varphi}_a(t,\mathbf{x})\right]\hat{\mathcal{P}} && \langle\!\langle \text{by eqs. (S.22) and (S.24)}\rangle\!\rangle\\
&= \hat{\mathcal{P}}\Big(D(D+1)\hat{\varphi}_a(t,\mathbf{x})\Big)\hat{\mathcal{P}} && \langle\!\langle \text{by eq. (S.23)}\rangle\!\rangle\\
&= D(D+1)\hat{\varphi}'_a(t,-\mathbf{x}),
\end{aligned} \tag{S.25}$$

and likewise

$$\left[\hat{\mathbf{J}}_+^2,\hat{\varphi}'_a(t,-\mathbf{x})\right] \;=\; C(C+1)\hat{\varphi}'_a(t,-\mathbf{x}). \tag{S.26}$$

Thus, the field multiplet $\hat{\varphi}'_a(x')$ has definite $(j_+,j_-)'=(D,C)$, which exchanges the j_+ and the j_- quantum numbers $(j_+,j_-)=(C,D)$ of the original multiplet $\hat{\varphi}_a(x)$.

Problem **2**(b):

First, consider Dirac equation. Rewriting eq. (10) as $\Psi'(\mathbf{x}',t')=\pm\gamma^0\Psi(\mathbf{x}=-\mathbf{x}',t=+t')$, we have

$$\begin{aligned}
(i\,\partial\!\!\!/'-m)\Psi'(x') &\equiv (i\gamma^0\partial_0+i\vec{\gamma}\cdot\nabla'-m)\times(\pm\gamma^0)\Psi(\mathbf{x}',t)\\
&= (\pm\gamma^0)(i\gamma^0\partial_0-i\vec{\gamma}\cdot\nabla'-m)\Psi(\mathbf{x}',t)\\
&= (\pm\gamma^0)(i\gamma^0\partial_0+i\vec{\gamma}\cdot\nabla-m)\Psi(-\mathbf{x},t)\\
&\equiv (\pm\gamma^0)(i\,\partial\!\!\!/-m)\Psi\Big|_{x'}.
\end{aligned} \tag{S.27}$$

Thus, $(i\,\partial\!\!\!/-m)\Psi(x)$ transforms under parity exactly as the Dirac field $\Psi(x)$ itself, which means that the Dirac equation is *covariant* under parity.

Now consider Dirac Lagrangian $\mathcal{L}\overline{\Psi}(i\,!\partial\!\!\!/-m)\Psi$. Conjugating eq. (10) we have $\overline{\Psi}'(\mathbf{x}',t')=\pm\overline{\Psi}(\mathbf{x}=-\mathbf{x}',t=+t')$, and hence in light of eq. (S.27),

$$\mathcal{L}'(x') \;=\; \overline{\Psi}'(x')(i\,\partial\!\!\!/'-m)\Psi'(x') \;=\; \overline{\Psi}(x)(i\,!\partial\!\!\!/-m)\Psi(x) \;=\; \mathcal{L}(x). \tag{S.28}$$

In other words, Dirac Lagrangian transforms under parity as a true scalar field, and consequently the Dirac Action $\int d^4x\mathcal{L}$ is invariant.

Problem $\mathbf{3}$(a):

As explained in class,

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\xi_s \end{pmatrix} \quad\Longrightarrow\quad \bar{u}(p,s) \;=\; \left(\xi_s^\dagger\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,,\,\xi_s^\dagger\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\right) \tag{S.29}$$

where ξ_s is the ordinary 3D 2–component spinor normalized to $\xi^\dagger\xi = 1$ and therefore

$$\sum_s (\xi_s\xi_s^\dagger) \;=\; 1 \tag{S.30}$$

as a 2×2 matrix. Consequently, in 4×4 matrix notations, we have

$$\begin{aligned}
\sum_s u(p,s)\bar{u}(p,s) \;&=\; \sum_s \left(\begin{array}{c|c} \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \\ \hline \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \end{array}\right) \\
&=\; \left(\begin{array}{c|c} \sqrt{E^2-(\mathbf{p}\boldsymbol{\sigma})^2} & (E-\mathbf{p}\boldsymbol{\sigma}) \\ \hline (E+\mathbf{p}\boldsymbol{\sigma}) & \sqrt{E^2-(\mathbf{p}\boldsymbol{\sigma})^2} \end{array}\right) \\
&=\; \left(\begin{array}{c|c} m & (E-\mathbf{p}\boldsymbol{\sigma}) \\ \hline (E+\mathbf{p}\boldsymbol{\sigma}) & m \end{array}\right) \;=\; m + \not{p}.
\end{aligned} \tag{S.31}$$

Likewise, for the negative-frequency spinors we have

$$v(p,s) \;=\; \begin{pmatrix} +\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \quad\Longrightarrow\quad \bar{v}(p,s) \;=\; \left(-\eta_s^\dagger\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,,\,+\eta_s^\dagger\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\right) \tag{S.32}$$

and therefore

$$\begin{aligned}
\sum_s v(p,s)\bar{v}(p,s) \;&=\; \sum_s \left(\begin{array}{c|c} -\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & +\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \\ \hline +\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \end{array}\right) \\
&=\; \left(\begin{array}{c|c} -\sqrt{E^2-(\mathbf{p}\boldsymbol{\sigma})^2} & (E-\mathbf{p}\boldsymbol{\sigma}) \\ \hline (E+\mathbf{p}\boldsymbol{\sigma}) & -\sqrt{E^2-(\mathbf{p}\boldsymbol{\sigma})^2} \end{array}\right) \\
&=\; \left(\begin{array}{c|c} -m & (E-\mathbf{p}\boldsymbol{\sigma}) \\ \hline (E+\mathbf{p}\boldsymbol{\sigma}) & -m \end{array}\right) \;=\; -m + \not{p}.
\end{aligned} \tag{S.33}$$

Problem **3**(b):

The constant spinors $u \equiv u(p,s)$ and $\bar{u}' \equiv \bar{u}(p',s')$ satisfy Dirac equations $\not{p}u = mu$ and $\bar{u}'\not{p}' = m\bar{u}'$. Applying both equations to the Dirac "sandwich" $\bar{u}'\gamma^\mu u$, we have

$$\bar{u}'\gamma^\mu u \;=\; \frac{1}{m}\,\bar{u}'\not{p}'\times\gamma^\mu u \;=\; \frac{1}{m}\,\bar{u}'\gamma^\mu\times\not{p}u \;=\; \frac{1}{2m}\,\bar{u}'(\not{p}'\gamma^\mu \;+\; \gamma^\mu\not{p})u. \tag{S.34}$$

Furthermore,

$$\begin{aligned}\not{p}'\gamma^\mu \;+\; \gamma^\mu\not{p} \;&\equiv\; p'_\nu\gamma^\nu\gamma^\mu \;+\; p_\nu\gamma^\mu\gamma^\nu \;=\; \tfrac{1}{2}(p'+p)_\nu\{\gamma^\mu,\gamma_\nu\} \;+\; \tfrac{1}{2}(p'-p)_\nu[\gamma^\nu,\gamma^\mu]\\ &=\; \tfrac{1}{2}(p'+p)_\nu\times 2g^{\mu\nu} \;+\; \tfrac{1}{2}(p'-p)_\nu\times 4iS^{\mu\nu}\end{aligned} \tag{S.35}$$

and therefore

$$\bar{u}'\gamma^\mu u \;=\; \frac{(p'+p)^\mu}{2m}\,\bar{u}'u \;+\; \frac{i(p'-p)_\nu}{m}\,\bar{u}'S^{\mu\nu}u. \tag{2}$$

$\mathcal{Q.E.D.}$

Problem **3**(c):

The negative-frequency spinors $v \equiv v(p,s)$ and $\bar{v}' \equiv \bar{v}(p',s')$ satisfy Dirac equations $\not{p}v = -mv$ and $\bar{v}'\not{p}' = -m\bar{v}'$. Consequently, proceeding exactly as above modulo signs, we have

$$\begin{aligned}\bar{u}'\gamma^\mu v \;&=\; \frac{(p'-p)^\mu}{2m}\,\bar{u}'v \;+\; \frac{i(p'+p)_\nu}{m}\,\bar{u}'S^{\mu\nu}v,\\ \bar{v}'\gamma^\mu u \;&=\; \frac{(-p'+p)^\mu}{2m}\,\bar{v}'u \;+\; \frac{i(-p'+p)_\nu}{m}\,\bar{v}'S^{\mu\nu}u,\\ \bar{v}'\gamma^\mu v \;&=\; \frac{(-p'-p)^\mu}{2m}\,\bar{v}'v \;+\; \frac{i(-p'+p)_\nu}{m}\,\bar{v}'S^{\mu\nu}v.\end{aligned} \tag{S.36}$$

Problem **3**(d):

Given $\mathbf{p}' = -\mathbf{p}$ and hence $E' = E = +\sqrt{\mathbf{p}^2+m^2}$, we have

$$v(p',s') \;=\; \begin{pmatrix} +\sqrt{E'-\mathbf{p}'\boldsymbol{\sigma}}\,\eta \\ -\sqrt{E'+\mathbf{p}\boldsymbol{\sigma}}\,\eta \end{pmatrix} \;=\; \begin{pmatrix} +\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\eta \\ -\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\eta \end{pmatrix} \tag{S.37}$$

where $\eta = \eta_{s'}$. At the same time,

$$u^\dagger(p,s) \;=\; \left(\xi^\dagger\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\middle|\,\xi^\dagger\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\right) \tag{S.38}$$

where $\xi = \xi_s$. Consequently,

$$u^\dagger(p,s)v(p',s') \;=\; \xi^\dagger\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\eta \;-\; \xi^\dagger\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\eta \;=\; 0, \quad \mathcal{Q}.\mathcal{E}.\mathcal{D}.$$