Problem **1**(a):

$\gamma^\mu\gamma^\nu = \pm\gamma^\nu\gamma^\mu$ where the sign is '+' for $\mu=\nu$ and '−' otherwise. Hence for any product Γ of the γ matrices, $\gamma^\mu\Gamma = (-1)^{n_\mu}\Gamma\gamma^\mu$ where n_μ is the number of $\gamma^{\nu\neq\mu}$ factors of Γ. For $\Gamma=\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, $n_\mu=3$ for any $\mu=0,1,2,3$; thus $\gamma^\mu\gamma^5=-\gamma^5\gamma^\mu$.

Problem **1**(b):

First,

$$\begin{aligned}\left(\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3\right)^\dagger &= -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = +i\gamma^3\gamma^2\gamma^1\gamma^0\\ &= +i((\gamma^3\gamma^2)\gamma^1)\gamma^0 = (-1)^3 i\,\gamma^0((\gamma^3\gamma^2)\gamma^1)\\ &= (-1)^{3+2}i\,\gamma^0(\gamma^1(\gamma^3\gamma^2)) = (-1)^{3+2+1}i\,\gamma^0(\gamma^1(\gamma^2\gamma^3))\\ &= +i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv +\gamma^5.\end{aligned} \tag{S.1}$$

Second,

$$\begin{aligned}(\gamma^5)^2 &= \gamma^5(\gamma^5)^\dagger = (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^3\gamma^2\gamma^1\gamma^0) = -\gamma^0\gamma^1\gamma^2(\gamma^3\gamma^3)\gamma^2\gamma^1\gamma^0\\ &= +\gamma^0\gamma^1(\gamma^2\gamma^2)\gamma^1\gamma^0 = -\gamma^0(\gamma^1\gamma^1)\gamma^0 = +\gamma^0\gamma^0 = +1.\end{aligned} \tag{S.2}$$

Problem **1**(c):

Any four distinct γ^κ, γ^λ, γ^μ, γ^ν are γ^0, γ^1, γ^2, γ^3 in some order. They all anticommute with each other, hence $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu=\epsilon^{\kappa\lambda\mu\nu}\gamma^0\gamma^1\gamma^2\gamma^3\equiv -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$. The rest is obvious.

Problem **1**(d):

$$\begin{aligned}i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5 &= \gamma_\kappa\,\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]}\\ &= \tfrac14\gamma_\kappa\left(\gamma^\kappa\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} - \gamma^{[\lambda)}\gamma^\kappa\gamma^{(\mu}\gamma^{\nu]} + \gamma^{[\lambda}\gamma^{\mu)}\gamma^\kappa\gamma^{(\nu]} - \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}\gamma^\kappa\right)\\ &= \tfrac14\left(4\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} + 2\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} + 4g^{[\lambda\mu}\gamma^{\nu]} + 2\gamma^{[\nu}\gamma^\mu\gamma^{\lambda]}\right)\\ &= \tfrac14(4+2+0-2)\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}.\end{aligned} \tag{S.3}$$

Problem $\mathbf{1}$(e):

Proof by inspection: In the Weyl basis, the 16 matrices are

$$\begin{aligned}
\mathbf{1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \\
i\gamma^{[i}\gamma^{j]} &= \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \qquad i\gamma^{[0}\gamma^{i]} = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & +i\sigma^i \end{pmatrix}, \\
\gamma^5\gamma^0 &= \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \qquad \gamma^5\gamma^1 = \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix},
\end{aligned} \tag{S.4}$$

and their linear independence is self-evident. Since there are only 16 independent 4×4 matrices altogether, any such matrix Γ is a linear combination of the matrices (S.4). $\mathcal{Q.E.D.}$

Algebraic Proof: Without making any assumption about the matrix form of the γ^μ operators, let us consider the Clifford algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$. Because of these anticommutation relations, one may re-order any product of the γ's as $\pm\gamma^0 \cdots \gamma^0\, \gamma^1 \cdots \gamma^1\, \gamma^2 \cdots \gamma^2\, \gamma^3 \cdots \gamma^3$ and then further simplify it to $\pm(\gamma^0 \text{ or } 1) \times (\gamma^1 \text{ or } 1) \times (\gamma^2 \text{ or } 1) \times (\gamma^3 \text{ or } 1)$. The net result is (up to a sign or $\pm i$ factor) one of the 16 operators 1, γ^μ, $i\gamma^{[\mu}\gamma^{\nu]}$, $-i\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \epsilon^{\lambda\mu\nu\rho}\gamma^5\gamma_\rho$ (*cf.* (d)) or $i\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = \epsilon^{\kappa\lambda\mu\nu}\gamma^5$ (*cf.* (c)). Consequently, any operator Γ algebraically constructed of the γ^μ's is a linear combination of these 16 operators.

Incidentally, the algebraic argument explains why the γ^μ (and hence all their products) should be realized as 4×4 matrices since any lesser matrix size would not accommodate 16 independent products. That is, the γ's are 4×4 matrices in four spacetime dimensions; different dimensions call for different matrix sizes. Specifically, in spacetimes of *even* dimensions d, there are 2^d independent products of the γ operators, so we need matrices of size $2^{d/2} \times 2^{d/2}$: 2×2 in two dimensions, 4×4 in four, 8×8 in six, 16×16 in eight, 32×32 in ten, *etc.*,*etc.*.

In odd dimensions, there are only 2^{d-1} independent operators because $\gamma^{d+1} \equiv (i)\gamma^0\gamma^1 \cdots \gamma^{d-1}$ — the analogue of the γ^5 operator in 4d — commutes rather than anticommutes with all the γ^μ and hence with the whole algebra. Consequently, one has two distinct representations of the Clifford algebra — one with $\gamma^{d+1} = +1$ and one with $\gamma^{d+1} = -1$ — but in each representation there are only 2^{d-1} independent operator products, which call for the matrix size of $2^{(d-1)/2} \times 2^{(d-1)/2}$. For example, in three spacetime dimensions (two space, one time), can take $(\gamma^0, \gamma^1, \gamma^2) = (\sigma_3, i\sigma_1, i\sigma_2)$ for $\gamma^4 \equiv i\gamma^0\gamma^1\gamma^2 = +1$ or $(\gamma^0, \gamma^1, \gamma^2) = (\sigma_3, i\sigma_1, -i\sigma_2)$ for $\gamma^4 = -1$, but in both cases we have 2×2 matrices. Likewise, we have 4×4 matrices in five dimensions, 8×8 in 7D, 16×16 in 9D, 32×32 in 11D, *etc.*, *etc.*

Problem **2**(a):

Despite anticommutativity of the fermionic fields, the Hermitian conjugation of an operator product reverses the order of operators without any extra sign factors, thus $(\Psi^\dagger_\alpha \Psi_\beta)^\dagger = +\Psi^\dagger_\beta \Psi_\alpha$. Consequently, for any 4×4 matrix Γ, $(\Psi^\dagger \Gamma \Psi^\dagger = +\Psi^\dagger \Gamma^\dagger \Psi$, and hence $(\overline{\Psi}\Gamma\Psi)^\dagger = \overline{\Psi}\overline{\Gamma}\Psi$ where $\overline{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$ is the Dirac conjugate of Γ.

Now consider the 16 matrices which appear in the bilinears (1). Obviously $\overline{1} = +1$ and this gives us $S^\dagger = +S$. We saw in class that $\overline{\gamma^\mu} = +\gamma^\mu$, and this gives us $(V^\mu)^\dagger = +V^\mu$. We also saw that $\overline{i\gamma^{[\mu}\gamma^{\nu]}} = -i\gamma^{[\nu}\gamma^{\mu]} = +i\gamma^{[\mu}\gamma^{\nu]}$, and this gives us $(T^{\mu\nu})^\dagger = +T^{\mu\nu}$. As to the γ^5 matrix, it is Hermitian (*cf.* 1.(b)) and anticommutes with γ^0, hence $\overline{\gamma^5} = \gamma^0(\gamma^5)^\dagger\gamma^0 = +\gamma^0\gamma^5\gamma^0 = -\gamma^5$ and therefore $\overline{i\gamma^5} = +i\gamma^5$, which gives us $P^\dagger = +P$. Finally, $\overline{\gamma^5\gamma^\mu} = \overline{\gamma^\mu}\,\overline{\gamma^5} = -\gamma^\mu\gamma^5 = +\gamma^5\gamma^\mu$, which gives us $(A^\mu)^\dagger = +A^\mu$. Thus, by inspection, all the bilinears (1) are Hermitian. $\mathcal{Q.E.D.}$

Problem **2**(b):

Under a continuous Lorentz symmetry $x \mapsto x' = Lx$, the Dirac spinor field and its conjugate transform according to

$$\Psi'(x') = M(L)\Psi(x = L^{-1}x'), \qquad \overline{\Psi}'(x') = \overline{\Psi}(x = L^{-1}x')M^{-1}(L), \tag{S.5}$$

hence any bilinear $\overline{\Psi}\Gamma\Psi$ transforms according to

$$\overline{\Psi}'(x')\Gamma\Psi(x') = \overline{\Psi}(x)\Gamma'\Psi(x) \tag{S.6}$$

where

$$\Gamma' = M^{-1}(L)\Gamma M(L). \tag{S.7}$$

Obviously, for $\Gamma = 1$, $\Gamma' = M^{-1}M = 1$. According to homework set #5 (problem 3(d)), for $\Gamma = \gamma^\mu$, $\Gamma' = M^{-1}\gamma^\mu M = L^\mu_{\ \nu}\gamma^\nu$. Similarly, $M^{-1}\gamma^\mu\gamma^\nu M = (M^{-1}\gamma^\mu M)(M^{-1}\gamma^\nu M) = L^\mu_{\ \kappa}\gamma^\kappa \times$

$L^{\nu}{}_{\lambda}\gamma^{\lambda}$ and hence for $\Gamma = \gamma^{[\mu}\gamma^{\nu]}$, $\Gamma' = L^{\mu}{}_{\kappa}L^{\nu}{}_{\lambda}\gamma^{[\kappa}\gamma^{\lambda]}$. Consequently,

$$S'(x') = S(x), \qquad V'^{\mu}(x') = L^{\mu}{}_{\nu}V^{\nu}(x), \qquad T'^{\mu\nu}(x') = L^{\mu}{}_{\kappa}L^{\nu}{}_{\lambda}T^{\kappa\lambda}(x), \tag{S.8}$$

which makes S a Lorentz scalar, V^{μ} a Lorentz vector and $T^{\mu\nu}$ a Lorentz tensor (with two antisymmetric indices).

The γ^5 matrix commutes with even products of the γ^{μ} matrices such as $\gamma^{\mu}\gamma^{\nu}$, hence it commutes with all $S^{\mu\nu}$ and therefore with $M(L) = \exp\bigl(-\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu}\bigr)$. Consequently, for $\Gamma = \gamma^5$, $\Gamma' = M^{-1}\gamma^5 M = \gamma^5$ while for $\Gamma = \gamma^5\gamma^{\mu}$, $\Gamma' = M^{-1}\gamma^5\gamma^{\mu}M = \gamma^5 M^{-1}\gamma^{\mu}M = \gamma^5(L^{\mu}{}_{\nu}\gamma^{\nu}) = L^{\mu}{}_{\nu}(\gamma^5\gamma^{\nu})$. Therefore,

$$P'(x') = P(x), \qquad A'^{\mu}(x') = L^{\mu}{}_{\nu}A^{\nu}(x), \tag{S.9}$$

which makes P a Lorentz scalar and A a Lorentz vector. $\mathcal{Q.E.D.}$

Problem **2**(c):

Under the parity symmetry $\mathcal{P}$, $(\mathbf{x}, t)' = (-\mathbf{x}, +t)$ and the Dirac field transforms according to

$$\Psi'(x') = \pm\gamma^0\Psi(x), \qquad \overline{\Psi}'(x') = \pm\overline{\Psi}(x)\gamma^0 \tag{S.10}$$

(*cf.* problem 2 of the previous set). Hence, parity properties of the Dirac bilinears (1) follow from the commutation relations of the 16 matrices 1.(e) with the γ^0 matrix. It is easy to verify that 1, γ^0, $\gamma^{[i}\gamma^{j]}$ and $\gamma^5\gamma^i$ commute with the γ^0 while γ^i, $\gamma^0\gamma^i$, $\gamma^5\gamma^0$ and γ^5 anticommute with the γ^0. Consequently,

- the S, V^0, T^{ij} and A^i remain invariant under parity, while
- the V^i, T^{0i}, A^0 and P change their signs.

In three-dimensional terms, this means that S and V^0 are true scalars, P and A^0 are pseudoscalars, $\mathbf{V}$ is a true or polar vector, $\mathbf{A}$ is a pseudovector or axial vector, and the tensor T contains one true vector T^{0i} and one axial vector $\frac{1}{2}\epsilon^{ijk}T^{jk}$. In space-time terms, we call S a (Lorentz) (true) scalar, P a (Lorentz) pseudoscalar, V^{μ} a (Lorentz) (true) vector and A^{μ} an (Lorentz) axial vector. Pedantically speaking, $T^{\mu\nu}$ is a Lorentz true tensor while $\widetilde{T}^{\kappa\lambda} \equiv \frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}T_{\mu\nu}$ is a Lorentz pseudotensor, but few people are that pedantic.

Problem **2**(d):

In the Weyl convention for the Dirac matrices, the charge conjugation symmetry $\mathcal{C}$ acts on the Dirac field according to $\Psi'(x) = \pm\gamma^2\Psi^*(x)$. Consequently

$$\Psi'^\dagger(x) \;=\; \mp\Psi^\top(x)\gamma^2 \quad\Longrightarrow\quad \overline{\Psi}'(x) \;=\; \Psi'^\dagger(x)\gamma^0 \;=\; \mp\Psi^\top(x)\gamma^2\gamma^0, \tag{S.11}$$

and therefore for any Dirac bilinear,

$$\overline{\Psi}'\Gamma\Psi' \;=\; -\Psi^\top\gamma^2\gamma^0\Gamma\gamma^2\Psi^* \;=\; +\Psi^\dagger(\gamma^2\gamma^0\Gamma\gamma^2)^\top\Psi \;=\; +\overline{\Psi}\gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2\Psi \;\equiv\; \overline{\Psi}\Gamma^c\Psi. \tag{S.12}$$

The second equality of this formula follows by transposition of the Dirac "sandwich" $\Psi^\top\cdots\Psi^*$, which carries an extra minus sign because the fermionic fields Ψ and Ψ^* anticommute with each other.

Problem **2**(e):

By inspection, $\mathbf{1}^c \equiv \gamma^0\gamma^2\gamma^0\gamma^2 = +\mathbf{1}$. The γ_5 matrix is symmetric and commutes with the $\gamma^0\gamma^2$, hence $\gamma_5^c = +\gamma_5$. Among the four γ_μ matrices, the γ_1 and γ_3 are anti-symmetric and commute with the $\gamma^0\gamma^2$ while the γ_0 and γ_2 are symmetric but anti-commute with the $\gamma^0\gamma^2$; hence, for all four γ_μ, $\gamma_\mu^c = -\gamma_\mu$. Finally, because of the transposition involved, $(\gamma_\mu\gamma_\nu)^c = \gamma_\nu^c\gamma_\mu^c = +\gamma_\nu\gamma_\mu$, hence $(\gamma^{[\mu}\gamma^{\nu]})^c = +\gamma^{[\nu}\gamma^{\mu]} = -\gamma^{[\mu}\gamma^{\nu]}$. Likewise, $(\gamma^5\gamma^\mu)^c = (\gamma^\mu)^c(\gamma^5)^c = -\gamma^\mu\gamma^6 = +\gamma^5\gamma^\mu$.

Therefore, according to eq. (S.12), the scalar S, the pseudoscalar P and the axial vector A_μ are C–even while the vector V_μ and the tensor $T_{\mu\nu}$ are C–odd.

Problem **3**(a):

Given the anticommutation relations (2), we have

$$\hat{a}_\alpha^\dagger\hat{a}_\beta\hat{a}_\delta \;=\; -\hat{a}_\alpha^\dagger\hat{a}_\delta\hat{a}_\beta \;=\; -(\delta_{\alpha,\delta} - \hat{a}_\delta\hat{a}_\alpha^\dagger)\hat{a}_\beta \;=\; +\hat{a}_\delta\hat{a}_\alpha^\dagger\hat{a}_\beta \;-\; \delta_{\alpha,\delta}\hat{a}_\beta \tag{S.13}$$

and therefore $[\hat{a}_\alpha^\dagger\hat{a}_\beta, \hat{a}_\delta] = -\delta_{\alpha,\delta}\hat{a}_\beta$.

Likewise,

$$\hat{a}_\alpha^\dagger\hat{a}_\beta\hat{a}_\gamma^\dagger \;=\; \hat{a}_\alpha^\dagger(\delta_{\beta,\gamma} - \hat{a}_\gamma^\dagger\hat{a}_\beta) \;=\; \delta_{\beta,\gamma}\hat{a}_\alpha^\dagger \;-\; \hat{a}_\alpha^\dagger\hat{a}_\gamma^\dagger\hat{a}_\beta \;=\; \delta_{\beta,\gamma}\hat{a}_\alpha^\dagger \;+\; \hat{a}_\gamma^\dagger\hat{a}_\alpha^\dagger\hat{a}_\beta \tag{S.14}$$

and therefore $[\hat{a}_\alpha^\dagger\hat{a}_\beta, \hat{a}_\gamma^\dagger] = +\delta_{\beta,\gamma}\hat{a}_\alpha^\dagger$.

Finally, by Leibniz rule

$$[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta] \;=\; [\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma]\hat{a}_\delta \;+\; \hat{a}^\dagger_\gamma [\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta] \;=\; +\delta_{\beta,\gamma} \hat{a}^\dagger_\alpha \hat{a}_\delta - \;\delta_{\alpha,\delta} \hat{a}^\dagger_\gamma \hat{a}_\beta \,. \tag{S.15}$$

Problem **3**(b):

According to eq. (S.15), the commutator $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]$ has exactly the same form as its bosonic counterpart. Hence, the proof of $[\hat{A}, \hat{B}] = \hat{C}$ proceeds exactly as in the bosonic case, *cf.* homework set #3 (problem 2(b)).

Problem **3**(c):

Using the Leibniz rules and eqs. (S.13) and (S.14),

$$[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta] \;=\; \delta_{\nu\alpha} \hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta \;+\; \delta_{\nu\beta} \hat{a}^\dagger_\mu \hat{a}^\dagger_\alpha \hat{a}_\gamma \hat{a}_\delta \;-\; \delta_{\mu\gamma} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\delta \;-\; \delta_{\mu\delta} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\nu \,. \tag{S.16}$$

Problem **3**(d):

Again, w have a fermionic analogue to the bosonic second-quantized operators we studied in homework set #3 (problem 2(d)). Given eqs. (4) and (S.16) (in which we exchange $\gamma \leftrightarrow \delta$), we have

$$\begin{aligned} \left[\hat{A}, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma\right] &= \sum_{\mu,\nu} \langle\mu| \hat{A}_1 |\nu\rangle \left[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma\right] \\ &= \sum_{\mu} \langle\mu| \hat{A}_1 |\alpha\rangle \; \hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma \; + \sum_{\mu} \langle\mu| \hat{A}_1 |\beta\rangle \; \hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\delta \hat{a}_\gamma \\ &\quad - \sum_{\nu} \langle\delta| \hat{A}_1 |\nu\rangle \; \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\gamma \; - \sum_{\nu} \langle\gamma| \hat{A}_1 |\nu\rangle \; \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\nu \end{aligned} \tag{S.17}$$

and consequently, in light of eq. (8),

$$\begin{aligned}
\left[\hat{A},\hat{B}\right] \;&=\; \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle \left[\sum_\mu \langle\mu|\,\hat{A}_1\,|\alpha\rangle\;\hat{a}^\dagger_\mu\hat{a}^\dagger_\beta\hat{a}_\delta\hat{a}_\gamma \;+\sum_\mu \langle\mu|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha\hat{a}^\dagger_\mu\hat{a}_\delta\hat{a}_\gamma \right.\\
&\qquad\qquad \left. -\sum_\nu \langle\delta|\,\hat{A}_1\,|\nu\rangle\;\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\nu\hat{a}_\gamma \;-\sum_\nu \langle\gamma|\,\hat{A}_1\,|\nu\rangle\;\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\delta\hat{a}_\nu\right]\\
&= \sum_{\mu,\beta,\gamma,\delta} \langle\mu\otimes\beta|\,\hat{A}_1(1^{\underline{\rm st}})\hat{B}_2\,|\gamma\otimes\delta\rangle\;\hat{a}^\dagger_\mu\hat{a}^\dagger_\beta\hat{a}_\delta\hat{a}_\gamma\\
&\quad + \sum_{\alpha,\mu,\gamma,\delta} \langle\alpha\otimes\mu|\,\hat{A}_1(2^{\underline{\rm nd}})\hat{B}_2\,|\gamma\otimes\delta\rangle\;\hat{a}^\dagger_\alpha\hat{a}^\dagger_\mu\hat{a}_\delta\hat{a}_\gamma\\
&\quad - \sum_{\alpha,\beta,\gamma,\nu} \langle\alpha\otimes\beta|\,\hat{B}_2\hat{A}_1(2^{\underline{\rm nd}})\,|\gamma\otimes\nu\rangle\;\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\nu\hat{a}_\gamma\\
&\quad - \sum_{\alpha,\beta,\nu,\delta} \langle\alpha\otimes\beta|\,\hat{B}_2\hat{A}_1(1^{\underline{\rm st}})\,|\nu\otimes\delta\rangle\;\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\delta\hat{a}_\nu\\
&\langle\!\langle \text{renaming indices} \rangle\!\rangle\\
&= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha\otimes\beta|\left[\left(A_1(1^{\underline{\rm st}})+A_1(2^{\underline{\rm nd}})\right),\hat{B}_2\right]|\gamma\otimes\delta\rangle\times\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\delta\hat{a}_\gamma\\
&= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha\otimes\beta|\,\hat{C}_2\,|\gamma\otimes\delta\rangle\times\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\delta\hat{a}_\gamma \;\equiv\; \hat{C}.
\end{aligned}$$

$\mathcal{Q}.\ \mathcal{E}.\ \mathcal{D}.$

Problem **4**.(a):

The simplest form of the Dirac Lagrangian is

$$\mathcal{L} \;=\; \overline{\Psi}(i\gamma^\mu\partial_\mu \,-\, m)\Psi, \tag{S.18}$$

which involves spacetime derivatives of the Ψ field but not of the $\overline{\Psi}$. Consequently, by Noether theorem

$$T_N^{\mu\nu} \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\times\partial^\nu\Psi \;-\; g^{\mu\nu}\mathcal{L} \;=\; \overline{\Psi}i\gamma^\mu\partial^\nu\Psi \;-\; g^{\mu\nu}\overline{\Psi}(i\gamma^\lambda\partial_\lambda - m)\Psi. \tag{S.19}$$

As usual for fields of non-zero spin, the Noether stress-energy tensor is not symmetric and the true stress-energy tensor is

$$T^{\mu\nu}_{\rm true} \;=\; T^{\mu\nu}_N \;+\; \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu} \tag{S.20}$$

for some three-index tensor $\mathcal{K}^{[\lambda\mu]\nu}$ antisymmetric in its first two indices (*cf.* problem 1 of the set 1). Fortunately, the correction (S.20) does not affect the net energy and momentum of the Dirac fields, thus

$$E_{\rm net} \;=\; \int d^3\mathbf{x}\, T^{00}_N(x) \;=\; \int d^3\mathbf{x}\, \overline{\Psi}(x)(-i\vec{\gamma}\cdot\nabla \;+\; m)\Psi(x) \tag{S.21}$$

(*cf.* eq. (5)), and

$$\mathbf{P}_{\rm net} \;=\; \int d^3\mathbf{x}\, T^{0\vec{\imath}}_N(x) \;=\; \int d^3\mathbf{x}\, \overline{\Psi}(x)(-i\gamma^0\nabla)\Psi(x). \tag{S.22}$$

Hence, in terms of the quantum Dirac fields $\hat{\Psi}(x)$ and $\hat{\Psi}^\dagger(x)$, the net mechanical momentum operator is

$$\hat{\mathbf{P}}_{\rm mech} \;=\; \int d^3\mathbf{x}\, \hat{\Psi}^\dagger(-i\nabla)\hat{\Psi}. \tag{9a}$$

At this point, let us expand the Dirac fields in terms of creation and annihilation operators. In the Schrödinger picture eqs. (5) become

$$\begin{aligned} \hat{\Psi}(\mathbf{x}) \;&=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{e^{+i\mathbf{p}\mathbf{x}}}{2E_{\mathbf{p}}} \sum_s \Big(u(\mathbf{p},s)\hat{a}_{\mathbf{p},s} \;+\; v(-\mathbf{p},s)\hat{b}^\dagger_{-\mathbf{p},s}\Big), \\ \hat{\Psi}(\mathbf{x}) \;&=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{e^{-i\mathbf{p}\mathbf{x}}}{2E_{\mathbf{p}}} \sum_{s'} \Big(u^\dagger(\mathbf{p},s')\hat{a}^\dagger_{\mathbf{p},s'} \;+\; v^\dagger(-\mathbf{p},s')\hat{b}_{-\mathbf{p},s'}\Big). \end{aligned} \tag{S.23}$$

Substituting these formulæ into eq. (9a) for the momentum operators gives

$$\begin{aligned} \hat{\mathbf{P}}_{\rm mech} &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\mathbf{p}}{(2E_{\mathbf{p}})^2} \sum_{s,s'} \Big(u^\dagger(\mathbf{p},s')\hat{a}^\dagger_{\mathbf{p},s'} \;+\; v^\dagger(-\mathbf{p},s')\hat{b}_{-\mathbf{p},s'}\Big) \times \Big(u(\mathbf{p},s)\hat{a}_{\mathbf{p},s} \;+\; v(-\mathbf{p},s)\hat{b}^\dagger_{-\mathbf{p},s}\Big) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\mathbf{p}}{2E_{\mathbf{p}}} \sum_s \Big(\hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s} \;+\; \hat{b}_{-\mathbf{p},s}\hat{b}^\dagger_{-\mathbf{p},s}\Big) \end{aligned} \tag{S.24}$$

where the second equality follows from

$$u^\dagger(\mathbf{p},s')u(\mathbf{p},s) \;=\; v^\dagger(-\mathbf{p},s')v(-\mathbf{p},s) \;=\; 2E_{\mathbf{p}}\,\delta_{s,s'} \tag{S.25}$$

while

$$u^\dagger(\mathbf{p},s')v(-\mathbf{p},s) \;=\; v^\dagger(-\mathbf{p},s')u(\mathbf{p},s) \;=\; 0. \tag{S.26}$$

Finally, we re-write the $\hat{b}\hat{b}^\dagger$ part of the last line of eq. (S.24) as

$$\begin{aligned}
\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\mathbf{p}}{2E_\mathbf{p}}\sum_s\Bigl(\hat{b}_{-\mathbf{p},s}\hat{b}^\dagger_{-\mathbf{p},s}\Bigr) \;&=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{-\mathbf{p}}{2E_\mathbf{p}}\sum_s\Bigl(\hat{b}_{+\mathbf{p},s}\hat{b}^\dagger_{+\mathbf{p},s}\Bigr)\\
&=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{-\mathbf{p}}{2E_\mathbf{p}}\sum_s\Bigl(-\hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s}\;+\;2E_\mathbf{p}(2\pi)^3\delta^{(3)}(\mathbf{0})\Bigr)\\
&=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_\mathbf{p}}\sum_s\Bigl(+\mathbf{p}\,\hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s}\Bigr)
\end{aligned} \tag{S.27}$$

where the last equality follows from

$$\int d^3\mathbf{p}\,\Bigl(\mathbf{p}\times\delta^{(3)}(\mathbf{0})\Bigr) \;=\; \mathbf{0} \tag{S.28}$$

by reasons of rotational symmetry. Consequently, substituting eq. (S.27) into eq. (S.24), we arrive at

$$\hat{\mathbf{P}}_{\rm mech} \;=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\mathbf{p}}{2E_\mathbf{p}}\sum_s\Bigl(\mathbf{p}\,\hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s}\;+\;\mathbf{p}\,\hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s}\Bigr). \tag{9b}$$

$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Problem **4**(b):

Electrons have charge $q=-e$, hence the gauge-covariant derivative of the electron field is

$$D_\mu\Psi(x) \;=\; \partial_\mu\Psi(x) \;-\; ieA_\mu(x)\Psi(x). \tag{S.29}$$

Consequently, the gauge invariant Lagrangian for the electron field Ψ coupled to the EM field $A_\mu(x)$ is

$$\begin{aligned}
\mathcal{L} \;&=\; \mathcal{L}_{\rm EM} \;+\; \overline{\Psi}(i\gamma^\mu D_\mu \;-\; m)\Psi\\
&=\; -\tfrac14 F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\gamma^\mu\partial_\mu \;-\; m)\Psi \;+\; eA_\mu\times\overline{\Psi}\gamma^\mu\Psi,
\end{aligned} \tag{S.30}$$

and hence the electric current

$$J^\mu \;=\; -\frac{\partial\mathcal{L}}{\partial A_\mu} \;=\; -e\overline{\Psi}\gamma^\mu\Psi. \tag{S.31}$$

Note that thus current is a true Lorentz vector and is odd under the charge conjugation symmetry $\mathcal{C}$ (*cf.* problem 2 of this set).

To be precise, eq. (S.31) presumes "classical" fermionic fields which anticommute with each other, thus the charge density J^0 can be written as

$$\begin{aligned} \text{either } J^0 \; - e\overline{\Psi}\gamma^0\Psi \;&=\; -e\Psi^\dagger\Psi \;\equiv\; -e\sum_\alpha \Psi^*_\alpha\Psi_\alpha \\ \text{or } J^0 \;&=\; +e\sum_\alpha \Psi_\alpha\Psi^*_\alpha\,. \end{aligned} \tag{S.32}$$

Alas, in the quantum theory $\hat{\Psi}^\dagger_\alpha\hat{\Psi}_\beta$ is *not* equal to $-\hat{\Psi}_\beta\hat{\Psi}^\dagger_\alpha$, and this gives rise to the operator ordering ambiguity in defining the quantum electric charge.

Fortunately, this ambiguity amounts to a constant. Indeed, the quantum Dirac fields $\hat{\Psi}^\dagger_\alpha(x)$ and $\hat{\Psi}_\alpha(x)$ are *linear* combinations of the fermionic creation and annihilation operators (*cf.* eq. (5)), and the latter either anticommute with each other or have c-number anticommutators (*cf.* eqs. (6) and (7)). Therefore, the anticommutator $\{\hat{\Psi}^\dagger_\alpha(x),\hat{\Psi}_\beta(y)\}$ is a c-number function of $(x-y)$. We shall calculate this function later in class, but for the moment all we need to know it's a c-number, and therefore

$$-e\sum_\alpha \hat{\Psi}^\dagger_\alpha(x)\hat{\Psi}_\alpha(x) \;=\; +e\sum_\alpha \hat{\Psi}_\alpha(x)\hat{\Psi}^\dagger_\alpha(x) \;+\; \text{a c-number constant.} \tag{S.33}$$

Consequently, however we order the creation and annihilation operators in the quantized electric charge operators, it will give us the same result up to a c-number constant (which may be infinite). Hence, we may just as well take the simplest ordering and allow for an extra constant term, thus

$$\hat{Q} \;=\; -e\int d^3\mathbf{x}\,\hat{\Psi}^\dagger(x)\hat{\Psi}(x) \;+\; \text{a c-number constant.} \tag{10a}$$

Next, let us expand the fields $\hat{\Psi}(x)$ and $\hat{\Psi}^\dagger(x)$ into creation and annihilation operators according to eqs. (S.23) and plug into the space integral (10a). Proceeding as in the previous question, we have

$$
\begin{aligned}
\int d^3\mathbf{x}\,\hat\Psi^\dagger\hat\Psi\ &= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{(2E_{\mathbf{p}})^2}\sum_{s,s'}\Bigl(u^\dagger(\mathbf{p},s')\hat a^\dagger_{\mathbf{p},s'}\ +\ v^\dagger(-\mathbf{p},s')\hat b_{-\mathbf{p},s'}\Bigr)\times\\
&\qquad\qquad\qquad\qquad\times\Bigl(u(\mathbf{p},s)\hat a_{\mathbf{p},s}\ +\ v(-\mathbf{p},s)\hat b^\dagger_{-\mathbf{p},s}\Bigr)\\
&= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\Bigl(\hat a^\dagger_{\mathbf{p},s}\hat a_{\mathbf{p},s}\ +\ \hat b_{-\mathbf{p},s}\hat b^\dagger_{-\mathbf{p},s}\Bigr)\\
&= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\Bigl(\hat a^\dagger_{\mathbf{p},s}\hat a_{\mathbf{p},s}\ +\ \hat b_{\mathbf{p},s}\hat b^\dagger_{\mathbf{p},s}\Bigr)\\
&= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\Bigl(\hat a^\dagger_{\mathbf{p},s}\hat a_{\mathbf{p},s}\ -\ \hat b^\dagger_{\mathbf{p},s}\hat b_{\mathbf{p},s}\ +\ 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{0})\Bigr)\\
&= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\Bigl(\hat a^\dagger_{\mathbf{p},s}\hat a_{\mathbf{p},s}\ -\ \hat b^\dagger_{\mathbf{p},s}\hat b_{\mathbf{p},s}\Bigr)\ +\ \text{infinite c-number constant,}
\end{aligned}
\tag{S.34}
$$

and therefore

$$
\hat Q\ = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\Bigl(-e\,\hat a^\dagger_{\mathbf{p},s}\hat a_{\mathbf{p},s}\ +\ e\,\hat b^\dagger_{\mathbf{p},s}\hat b_{\mathbf{p},s}\Bigr)\ +\ C \tag{S.35}
$$

where C is the sum of c-number constants from eqs. (10.a) and (S.34).

To determine the value of C, note that the vacuum state of the theory is invariant under the charge conjugation symmetry and therefore must have zero electric charge, $\hat Q\,|0\rangle = 0$. On the other hand, the vacuum state $|0\rangle$ is annihilated by all the $\hat a_{\mathbf{p},s}$ and $\hat b_{\mathbf{p},s}$ operators and hence by the terms on the right hand side of eq. (S.35) except for the constant C. Consequently, $\hat Q\,|0\rangle = C\,|0\rangle$ and the electric charge of the vacuum is C. And since this charge must vanish, we must have $C = 0$ — *i.e.*, somehow the constant terms in eqs. (10a) and (S.34) must cancel each other — and therefore

$$
\hat Q\ = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\Bigl(-e\,\hat a^\dagger_{\mathbf{p},s}\hat a_{\mathbf{p},s}\ +\ e\,\hat b^\dagger_{\mathbf{p},s}\hat b_{\mathbf{p},s}\Bigr). \tag{10b}
$$

$\mathcal{Q.E.D.}$

Problem **4**(c):

In this question, we start with eq. (11) for the net spin operator as a space integral of a Dirac bilinear, so our first step is to expand the fields into momentum modes (S.23) and plug the expansion into eq. (11). This gives us

$$\hat{\mathbf{S}}_{\rm net} \;=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,\tilde{\mathbf{S}}_{\mathbf{p}} \tag{S.36}$$

where

$$\tilde{\mathbf{S}}_{\mathbf{p}} \;=\; \frac{1}{2E_{\mathbf{p}}}\sum_{s,s'}\Big(u^\dagger(\mathbf{p},s)\hat{a}^\dagger_{\mathbf{p},s} \,+\, v^\dagger(-\mathbf{p},s)\hat{b}_{-\mathbf{p},s}\Big)\mathbf{S}\Big(u(\mathbf{p},s')\hat{a}_{\mathbf{p},s'} \,+\, v(-\mathbf{p},s')\hat{b}^\dagger_{-\mathbf{p},s'}\Big). \tag{S.37}$$

Next, we need to evaluate the Dirac "sandwiches" $u^\dagger\mathbf{S}u$, $v^\dagger\mathbf{S}v$, *etc.*, where

$$\mathbf{S} \;=\; \begin{pmatrix} \frac12\boldsymbol{\sigma} & 0 \\ 0 & \frac12\boldsymbol{\sigma} \end{pmatrix}. \tag{S.38}$$

For the non-relativistic modes ($|\mathbf{p}|\ll m$) we approximate

$$u(\mathbf{p},s) \;\approx\; u(\mathbf{0},s) \;=\; \sqrt{m}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}, \qquad v(-\mathbf{p},s) \;\approx\; v(\mathbf{0},s) \;=\; \sqrt{m}\begin{pmatrix} \eta_s \\ \eta_s \end{pmatrix}, \tag{S.39}$$

which gives us

$$\begin{aligned} u^\dagger(\mathbf{p},s)\mathbf{S}u(\mathbf{p},s') \;&\approx\; m\,\xi^\dagger_s\boldsymbol{\sigma}\xi_{s'}\,,\\ v^\dagger(-\mathbf{p},s)\mathbf{S}v(-\mathbf{p},s') \;&\approx\; m\,\eta^\dagger_s\boldsymbol{\sigma}\eta_{s'}\,,\\ u^\dagger(\mathbf{p},s)\mathbf{S}v(-\mathbf{p},s') \;&=\; O(|\mathbf{p}|) \;\approx\; 0,\\ v^\dagger(-\mathbf{p},s)\mathbf{S}u(\mathbf{p},s') \;&=\; O(|\mathbf{p}|) \;\approx\; 0, \end{aligned} \tag{S.40}$$

and consequently

$$\tilde{\mathbf{S}}_{\mathbf{p}} \;\approx\; \sum_{s,s'}\Big(\xi^\dagger_s\frac{\boldsymbol{\sigma}}{2}\xi_{s'}\times\hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s'} \;+\; \eta^\dagger_s\frac{\boldsymbol{\sigma}}{2}\eta_{s'}\times\hat{b}_{-\mathbf{p},s}\hat{b}^\dagger_{-bp,s'}\Big) \;+\; O(|\mathbf{p}|/m). \tag{S.41}$$

At this point, let us separate the $\hat{a}^\dagger\hat{a}$ terms from the $\hat{b}\hat{b}^\dagger$ terms in the momentum integral (S.36), and then in the $\hat{b}\hat{b}^\dagger$ part change the sign of the integration variable $\mathbf{p}$. Putting the two parts back together now gives us

$$\hat{\mathbf{S}}_{\rm net} \;=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,\hat{\mathbf{S}}_{\mathbf{p}} \tag{12}$$

where *for the non-relativistic momenta*

$$\hat{\mathbf{S}}\;\approx\;\sum_{s,s'}\Bigl(\xi_s^\dagger\frac{\boldsymbol{\sigma}}{2}\xi_{s'}\times\hat a^\dagger_{\mathbf{p},s}\hat a_{\mathbf{p},s'}\;+\;\eta_s^\dagger\frac{\boldsymbol{\sigma}}{2}\eta_{s'}\times\hat b_{\mathbf{p},s}\hat b^\dagger_{bp,s'}\Bigr)\;+\;O(|\mathbf{p}|/m). \tag{S.42}$$

To re-write this formula in the form of eq. (13), we note that the two-component spinors ξ_s and η_s have opposite spins. Specifically, $\eta_s=\xi_{-s}=\sigma_2\xi_s^*$ and therefore

$$\eta_s^\dagger\frac{\boldsymbol{\sigma}}{2}\eta_{s'}\;=\;\xi_s^\top\sigma_2\frac{\boldsymbol{\sigma}}{2}\sigma_2\xi_{s'}^*\;=\;\xi_{s'}^\dagger\left(\sigma_2\frac{\boldsymbol{\sigma}}{2}\sigma_2\right)^\top\xi_s\;=\;-\xi_{s'}^\dagger\frac{\boldsymbol{\sigma}}{2}\xi_s\,. \tag{S.43}$$

At the same time,

$$\hat b_{\mathbf{p},s}\hat b^\dagger_{\mathbf{p},s'}\;=\;-\hat b^\dagger_{\mathbf{p},s'}\hat b_{\mathbf{p},s}\;+\;2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{0})\times\delta_{s,s'}\,. \tag{S.44}$$

Consequently, we may re-write the $\hat b\hat b^\dagger$ part of eq. (S.42) as

$$\sum_{s,s'}\eta_s^\dagger\frac{\boldsymbol{\sigma}}{2}\eta_{s'}\times\hat b_{\mathbf{p},s}\hat b^\dagger_{bp,s'}\;=\;+\sum_{s,s'}\xi_{s'}^\dagger\frac{\boldsymbol{\sigma}}{2}\xi_s\times\hat b^\dagger_{\mathbf{p},s'}\hat b_{\mathbf{p},s}\;-\;2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{0})\times\sum_s\xi_s^\dagger\frac{\boldsymbol{\sigma}}{2}\xi_s \tag{S.45}$$

where the second term on the right hand side vanishes because

$$\sum_s\xi_s^\dagger\frac{\boldsymbol{\sigma}}{2}\xi_s\;\equiv\;\mathrm{tr}\,\frac{\boldsymbol{\sigma}}{2}\;=\;0. \tag{S.46}$$

Hence, interchanging the summation spin indices s and s', we have

$$\sum_{s,s'}\eta_s^\dagger\frac{\boldsymbol{\sigma}}{2}\eta_{s'}\times\hat b_{\mathbf{p},s}\hat b^\dagger_{bp,s'}\;=\;+\sum_{s,s'}\xi_s^\dagger\frac{\boldsymbol{\sigma}}{2}\xi_{s'}\times\hat b^\dagger_{\mathbf{p},s}\hat b_{\mathbf{p},s'}\,, \tag{S.47}$$

and plugging this formula into eq. (S.45) finally gives us

$$\tilde{\mathbf{S}}_{\mathbf{p}}\;\approx\;\sum_{s,s'}\xi_s^\dagger\frac{\boldsymbol{\sigma}}{2}\xi_{s'}\times\left(\hat a^\dagger_{\mathbf{p},s}\hat a_{\mathbf{p},s'}\;+\;\hat b^\dagger_{\mathbf{p},s}\hat b_{\mathbf{p},s}\right)\;+\;O(|\mathbf{p}|/m) \tag{13}$$

for the non-relativistic modes $\mathbf{p}$. $\mathcal{Q.E.D}$.