

In the center-of-mass frame, the partial cross-section for a two particles \rightarrow two particles process $A + B \rightarrow 1 + 2$ is calculated in the Peskin & Schroeder textbook in eq. (4.84):

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} &= \frac{1}{2E_A 2E_B |v_A - v_B|} \times \frac{|\mathbf{p}_1|}{16\pi^2 E_{\text{c.m.}}} \times |\mathcal{M}(A + B \rightarrow 1 + 2)|^2 & (4.84) \\ &= \frac{|\mathbf{p}_1|}{|\mathbf{p}_A|} \times \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{c.m.}}^2}. & (S.1) \end{aligned}$$

When one or both of the final-state particles have spin but its polarization is not measured in the experiment, one should sum the $|\mathcal{M}|^2$ over the final particles' spins. Likewise, when the initial particles have spins but the colliding beams are un-polarized — that is, the initial particles are equally likely to have any particular spin state — one should average the $|\mathcal{M}|^2$ over all such spin states. Thus, for the $e^+ + e^- \rightarrow H^0 + Z^0$ process at hand, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{|\mathbf{p}_Z|}{|\mathbf{p}_e|} \times \frac{1}{64\pi^2 E_{\text{c.m.}}^2} \times \left(\overline{|\mathcal{M}|^2} \stackrel{\text{def}}{=} \frac{1}{2} \sum_{s(e^+)} \frac{1}{2} \sum_{s(e^-)} \sum_{s(Z^0)} |\mathcal{M}|^2 \right), \quad (S.2)$$

and according to eq. (4) of the problem set,

$$\overline{|\mathcal{M}(e^+ + e^- \rightarrow H^0 + Z^0)|^2} = A \left(\frac{e^2 M_{Z^0}}{E_{\text{c.m.}}^2 - M_{Z^0}^2} \right)^2 \times \left((p_{e^+} \cdot p_{e^-}) + \frac{2}{M_{Z^0}^2} (p_{Z^0} \cdot p_{e^+})(p_{Z^0} \cdot p_{e^-}) \right). \quad (4)$$

In terms of the center-of-mass-frame energies and momenta, $\mathbf{p}_{e^+} = -\mathbf{p}_{e^-}$, $E_{e^+} = E_{e^-} = \frac{1}{2}E_{\text{c.m.}}$, we have

$$(p_{e^+} \cdot p_{e^-}) = E_{e^-}^2 + \mathbf{p}_{e^-}^2 \approx 2E_{e^-}^2 = \frac{1}{2}E_{\text{c.m.}}^2. \quad (S.3)$$

Also,

$$(p_{Z^0} \cdot p_{e^\pm}) = E_{Z^0} E_{e^\pm} - \mathbf{p}_{Z^0} \cdot \mathbf{p}_{e^\pm} = \frac{1}{2}E_{\text{c.m.}} (E_{Z^0} \pm |\mathbf{p}_{Z^0}| \cos \theta) \quad (S.4)$$

where θ is the angle between the directions of the initial electron and the final Z^0 particle — or

equivalently, between the initial positron and the final Higgs particle — and consequently

$$(p_{Z^0} \cdot p_{e^+})(p_{Z^0} \cdot p_{e^-}) = \frac{1}{4}E_{\text{c.m.}}^2 (E_{Z^0}^2 - \mathbf{p}_{Z^0}^2 \cos^2 \theta) = \frac{1}{4}E_{\text{c.m.}}^2 (E_{Z^0}^2 \sin^2 \theta + M_{Z^0}^2 \cos^2 \theta). \quad (\text{S.5})$$

Combining eqs. (4), (S.3) and (S.5) together, we have

$$\overline{|\mathcal{M}(e^+ + e^- \rightarrow H^0 + Z^0)|^2} = \frac{\frac{1}{2}Ae^4 E_{\text{c.m.}}^2 M_{Z^0}^2}{(E_{\text{c.m.}}^2 - M_{Z^0}^2)^2} \times \left(2 + \frac{\mathbf{p}_{Z^0}^2}{M_{Z^0}^2} \sin^2 \theta \right) \quad (\text{S.6})$$

and consequently,

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{|\mathbf{p}_{Z^0}|}{|\mathbf{p}_e|} \times \frac{A\alpha_{EM}^2 M_{Z^0}^2}{8(E_{\text{c.m.}}^2 - M_{Z^0}^2)^2} \times \left(2 + \frac{\mathbf{p}_{Z^0}^2}{M_{Z^0}^2} \sin^2 \theta \right) \quad (\text{S.7})$$

The last factor here governs the angular distribution of the Z^0 and Higgs particles produced in the electron-positron collision. The exact shape of this distribution depends on the collision's energy via the $\mathbf{p}_{Z^0}^2/M_{Z^0}^2$ coefficient, so our next task is to calculate the $\mathbf{p}_{Z^0}^2 = \mathbf{p}_H^2$.

The energies of the Z^0 particle and the Higgs particle produced in the $e^+ + e^-$ collision obey the following equations:

$$\begin{aligned} E_{Z^0} + E_H &= E_{\text{c.m.}}, \\ \mathbf{p}_{Z^0} + \mathbf{p}_H &= \mathbf{0}, \\ E_{Z^0}^2 - E_H^2 &= (\mathbf{p}_{Z^0}^2 + M_{Z^0}^2) - (\mathbf{p}_H^2 + M_H^2) = M_{Z^0}^2 - M_H^2, \\ E_{Z^0} - E_H &= \frac{E_{Z^0}^2 - E_H^2}{E_{Z^0} + E_H} = \frac{M_{Z^0}^2 - M_H^2}{E_{\text{c.m.}}}, \end{aligned} \quad (\text{S.8})$$

and hence

$$E_{Z^0} = \frac{E_{\text{c.m.}}^2 + M_{Z^0}^2 - M_H^2}{2E_{\text{c.m.}}}, \quad E_H = \frac{E_{\text{c.m.}}^2 + M_H^2 - M_{Z^0}^2}{2E_{\text{c.m.}}} \quad (\text{S.9})$$

while

$$\begin{aligned} \mathbf{p}_{Z^0}^2 = \mathbf{p}_H^2 &= E_{Z^0}^2 - M_{Z^0}^2 = E_H^2 - M_H^2 \\ &= \frac{1}{4E_{\text{c.m.}}^2} (E_{\text{c.m.}}^4 - 2E_{\text{c.m.}}^2(M_{Z^0}^2 + M_H^2) + (M_{Z^0}^2 - M_H^2)^2) \\ &= \frac{1}{4E_{\text{c.m.}}^2} (E_{\text{c.m.}}^2 - (M_H + M_{Z^0})^2) (E_{\text{c.m.}}^2 - (M_H - M_{Z^0})^2). \end{aligned} \quad (\text{S.10})$$

The highest total energies available at the LEP II accelerator were barely above the $Z^0 + \text{Higgs}$ threshold — and that's assuming the Higgs is as light as it could possibly be in light of current

experimental data. In other words,

$$E_{\text{c.m.}} = E_{Z^0} + E_H = M_{Z^0} + M_H + \epsilon \quad (\text{S.11})$$

for a rather small ϵ (energy above the threshold), which means that the final-state particles must have low (non-relativistic) momenta. Indeed according to eq. (S.10), in this regime $\mathbf{p}_{Z^0}^2 \approx 2\epsilon M_{Z^0} M_H / (M_{Z^0} + M_H) \ll M_{Z^0}^2$, and hence according to eq. (S.7) the angular distribution of the final-state particles is approximately isotropic.

At higher collision energies — hopefully available at future accelerators — the Z^0 particle would be relativistic and hence more likely to fly away perpendicular to the $e^+ + e^-$ collision axis than along it. Ultimately, in the extremely high energy limit we have $\mathbf{p}_{Z^0}^2 \approx (\frac{1}{2}E_{\text{c.m.}})^2 \gg M_{Z^0}^2$ and the partial cross section (S.7) is proportional to $\sin^2 \theta$. In this regime, the angular distribution of the Z^0 particles — and hence the Higgs particles as well — is similar to the angular distribution of the EM radiation of an electric dipole. The reason for this similarity is that the Z^0 and the photon are both vector particles (spin=1) with similar couplings to electrons — and the ultra-relativistic Z^0 particles are in effect approximately massless.

Finally, consider the total cross-section σ of the $e^+ + e^- \rightarrow Z^0 + H$ process as a function of the collision energy. Integrating eq. (S.7) over the solid angle $d\Omega$ and substituting eq. (S.10) for the $|\mathbf{p}_{Z^0}|$, we arrive at

$$\sigma_{\text{tot}}(e^+ + e^- \rightarrow Z^0 + H) = \frac{\pi A \alpha_{EM}^2}{12} \times \frac{E_{\text{c.m.}}^2}{(E_{\text{c.m.}}^2 - M_{Z^0}^2)^2} \times \left(\beta^3 + \frac{12\beta M_{Z^0}^2}{E_{\text{c.m.}}^2} \right) \quad (\text{S.12})$$

where

$$\beta \equiv \frac{|\mathbf{p}_{Z^0}|}{\frac{1}{2}E_{\text{c.m.}}} = \sqrt{1 - \left(\frac{M_{Z^0} + M_H}{E_{\text{c.m.}}} \right)^2} \sqrt{1 - \left(\frac{M_{Z^0} - M_H}{E_{\text{c.m.}}} \right)^2}. \quad (\text{S.13})$$

Near the energy threshold (S.11), the cross-section

$$\sigma_{\text{tot}}(e^+ + e^- \rightarrow Z^0 + H) \approx \frac{2\pi\sqrt{2}A\alpha_{EM}^2 M_{Z^0}^{5/2} \epsilon^{1/2}}{M_N^{3/2} (M_H + M_{Z^0})^{3/2} (M_H + 2M_{Z^0})^2} \quad (\text{S.14})$$

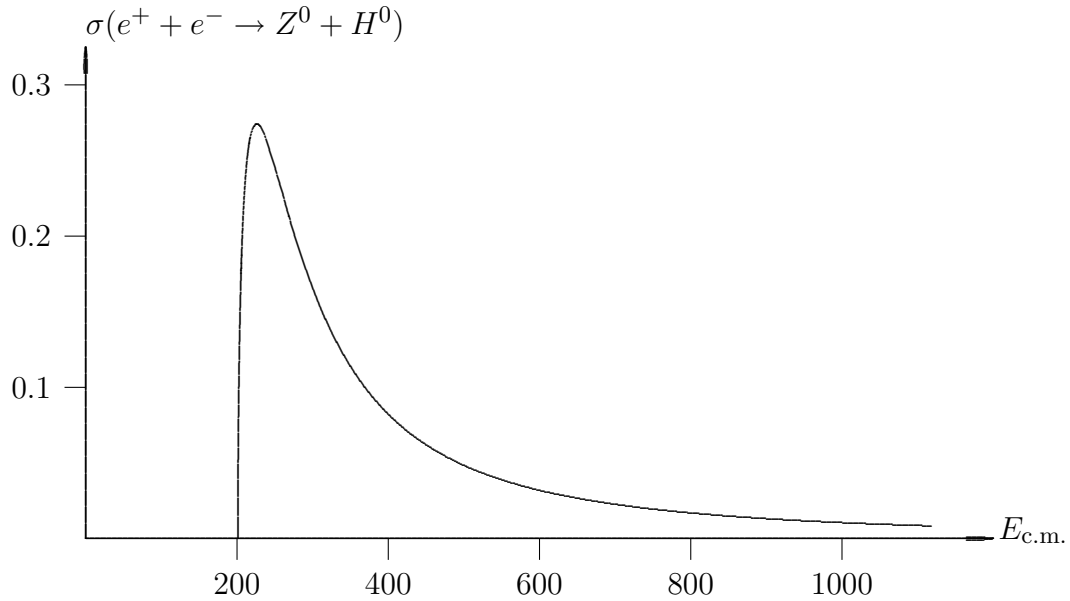
raises with energy as $\sqrt{\epsilon}$, but not too far above the threshold it reaches its maximal value and

starts decreasing. At very high energies, we have

$$\sigma(e^+ + e^- \rightarrow Z^0 + H) \approx \frac{\pi A \alpha_{EM}^2}{12 E_{c.m.}^2} \quad (\text{S.15})$$

and the cross-section decreases with the energy as $1/E_{c.m.}^2$.

For generic energies, eq. (S.12) is too complicated to study analytically. Instead, let me simply show you the numeric plot:



On this plot, $E_{c.m.}$ is in GeV, the cross-section σ is in picobarns ($1 \text{ pb} = 10^{-36} \text{ cm}^2$) and the Higgs mass is assumed to be $M_H = 110 \text{ GeV}$, which is a bit heavier than the $M_{Z^0} = 91.4 \text{ GeV}$.