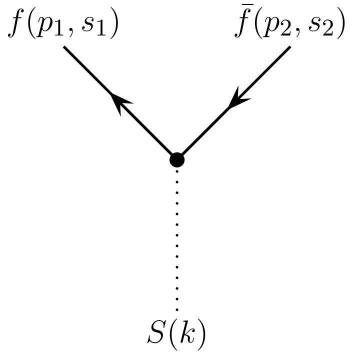
Problem **1**:

At the tree level, the decay $S \to f + \bar{f}$ follows from a single Feynman diagram



$$\text{hence} \quad i\mathcal{M} \;=\; \bar{u}(p_1, s_1)(-ig)v(p_2, s_2). \tag{S.1}$$

Note that this amplitude refers to specific spin states of the final fermion and antifermion. For the purpose of calculating the net (*i.e.*, un-polarized) decay rate, we need to sum the $|\mathcal{M}|^2$ over the final state spins. Thus

$$\begin{aligned}
\sum_{s_1,s_2} |\mathcal{M}|^2 \;&=\; g^2 \sum_{s_1,s_2} \bar{u}(p_1, s_1)v(p_2, s_2) \times \bar{v}(p_2, s_2)u(p_2, s_1) \\
&=\; g^2 \,\mathrm{tr}\left(\left(\sum_{s_1} u(p_1, s_1)\bar{u}(p_1, s_1)\right)\left(\sum_{s_2} v(p_2, s_2)\bar{v}(p_2, s_2)\right)\right) \\
&=\; g^2 \;\mathrm{tr}\Big((\not{p}_1 + m_f)\,(\not{p}_2 - m_f)\Big) \\
&=\; g^2\Big(\mathrm{tr}(\not{p}_1\,\not{p}_2) \;-\; m_f^2\,\mathrm{tr}(1)\Big) \;=\; g^2\Big(4p_1p_2 \;-\; 4m_f^2\Big) \\
&=\; g^2\Big(2(p_1 + p_2 = k)^2 \;-\; 8m_f^2\Big) \;=\; 2g^2(M_s^2 \;-\; 4m_f^2).
\end{aligned} \tag{S.2}$$

Finally, for the two-body decay, the phase-space factor evaluates to

$$\frac{1}{2M_s}\int\frac{d^3\mathbf{p}_1}{(2\pi)^3\,2E_1}\int\frac{d^3\mathbf{p}_2}{(2\pi)^3\,2E_2}\,(2\pi^4)\delta^{(4)}(p_1 + p_2 - k) \;=\; \frac{|\mathbf{p}_1||}{8\pi M_s^2} \tag{S.3}$$

where

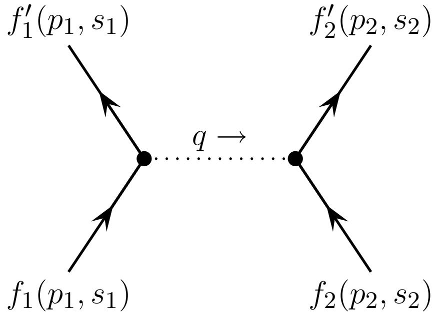
$$|\mathbf{p}_1| \;=\; |\mathbf{p}_2| \;=\; \sqrt{(\tfrac{1}{2}M_s)^2 \;-\; m_f^2}\,. \tag{S.4}$$

Putting all these factors together, we arrive at

$$\Gamma(S \to f + \bar{f}) \;=\; \frac{|\mathbf{p}_1||}{8\pi M_s^2} \times \sum_{s_1,s_2} |\mathcal{M}|^2 \;=\; \frac{g^2 M_s}{8\pi} \times \left(1 \,-\, \frac{4m_f^2}{M_s^2}\right)^{3/2}. \tag{S.5}$$

Problem **2**:

As discussed in class, scattering of fermions in the Yukawa theory proceeds by exchange of virtual scalar quanta between the fermions. For the problem at hand, the two fermions are distinct rather than identical, hence only one tree-level Feynman diagram contributes to the scattering:



$$q = p_1 - p_1' = p_2' - p_2\,, \tag{S.6}$$

and therefore

$$\mathcal{M}_{\rm tree}(f_1 + f_2 \to f_1 + f_2) \;=\; -\frac{g_1 g_2}{q^2 - M_s^2} \times \bar{u}(p_1', s_1') u(p_1, s_1) \times \bar{u}(p_2', s_2') u(p_2, s_2). \tag{S.7}$$

For specific spin states of all particles, the partial cross section of elastic scattering is given by

$$\frac{d\sigma}{d\Omega_{\rm c.m.}} \;=\; \frac{|\mathcal{M}|^2}{64\pi^2 E_{\rm c.m.}^2}\,, \tag{S.8}$$

but for un-polarized beams and spin-blind detectors we should sum this formula over the final particles' spins and average over spins of the initial particles, thus

$$\begin{aligned} \frac{d\sigma}{d\Omega_{\rm c.m.}} \;&=\; \frac{1}{64\pi^2 E_{\rm c.m.}^2} \times \tfrac{1}{2}\sum_{s_1} \tfrac{1}{2}\sum_{s_2} \sum_{s_1'} \sum_{s_2'} |\mathcal{M}|^2 \qquad\qquad (\text{S.9}) \\ &=\; \frac{1}{256\pi^2 E_{\rm c.m.}^2} \times \left(\frac{g_1 g_2}{q^2 - M_s^2}\right)^2 \times \sum_{s_1,s_1'} \left|\bar{u}(p_1', s_1') u(p_1, s_1)\right|^2 \times \sum_{s_2,s_2'} \left|\bar{u}(p_2', s_2') u(p_2, s_2)\right|^2. \end{aligned}$$

The spin sums on the second line here are similar to the spin sum in eq. (S.2):

$$\begin{aligned}\sum_{s_1,s_1'}\left|\bar{u}(p_1',s_1')u(p_1,s_1)\right|^2 &= \operatorname{tr}\left(\left(\sum_{s_1}u(p_1,s_1)\bar{u}(p_1,s_1)\right)\left(\sum_{s_1'}u(p_1',s_1')\bar{u}(p_1',s_1')\right)\right)\\ &= \operatorname{tr}\Big((\not{p}_1+m_1)\,(\not{p}_1'+m_1)\Big) \qquad\qquad (\text{S.10})\\ &= 4p_1p_1' + 4m_1^2 = 8m_1^2 - 2(p_1-p_1')^2\\ &= 8m_1^2 - 2q^2\end{aligned}$$

and likewise

$$\sum_{s_2,s_2'}\left|\bar{u}(p_2',s_2')u(p_2,s_2)\right|^2 = 8m_2^2 - 2q^2,$$

therefore

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{g_1^2g_2^2}{64\pi^2E_{\text{c.m.}}^2}\times\frac{(4m_1^2-q^2)(4m_2^2-q^2)}{(q^2-M_s^2)^2}\,. \qquad (\text{S.11})$$

Finally, we should integrate over the scattered particles' directions and calculate the total cross-section. In the center-of-mass frame, $q^0=0$, $\mathbf{q}^2=(\mathbf{p}_1-\mathbf{p}_1')^2=2\mathbf{p}_1^2(1-\cos\theta)$, hence

$$d\Omega = 2\pi\,d(-\cos\theta) = \frac{2\pi}{2\mathbf{p}^2}\,d\mathbf{q}^2. \qquad (\text{S.12})$$

Consequently, substituting $q^2=-\mathbf{q}^2$ in eq. (S.11) and integrating over $d\mathbf{q}^2$, we arrive at

$$\begin{aligned}\sigma_{\text{tot}} &= \frac{g_1^2g_2^2}{64\pi^2E_{\text{c.m.}}^2}\times\frac{2\pi}{2\mathbf{p}^2}\int\limits_0^{4\mathbf{p}^2}d\mathbf{q}^2\,\frac{(4m_1^2+\mathbf{q}^2)(4m_2^2+\mathbf{q}^2)}{(M_s^2+\mathbf{q}^2)^2} \qquad (\text{S.13})\\ &= \frac{g_1^2g_2^2}{16\pi E_{\text{c.m.}}^2}\left[1+\frac{(4m_1^2-M_s^2)(4m_2^2-M_s^2)}{M_s^2(M_s^2+4\mathbf{p}^2)}+\frac{2m_1^2+2m_2^2-M_s^2}{2\mathbf{p}^2}\log\frac{M_s^2+4\mathbf{p}^2}{M_s^2}\right]\end{aligned}$$

where

$$\mathbf{p}^2 = \tfrac{1}{4}E_{\text{c.m.}}^2 - \tfrac{1}{2}(m_1^2+m_2^2) + \frac{(m_1^2-m_2^2)^2}{4E_{\text{c.m.}}^2} \qquad (\text{S.14})$$

is the solution of the kinematical relation

$$E_{\text{c.m.}} = E_1 + E_2 = \sqrt{m_1^2+\mathbf{p}^2} + \sqrt{m_2^2+\mathbf{p}^2}.$$

Problem **3**(a):

We begin with the muon decay amplitude

$$\mathcal{M}(\mu^- \to e^- \nu_\mu \bar{\nu}_e) \;=\; \frac{G_F}{\sqrt{2}}\,\left[\bar{u}(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\right] \times \left[\bar{u}(e^-)(1-\gamma^5)\gamma_\alpha v(\bar{\nu}_e)\right], \tag{3}$$

Its complex conjugate can be written as

$$\mathcal{M}^* \;=\; \frac{G_F}{\sqrt{2}}\,\left[\bar{u}(\mu^-)\gamma^\beta(1+\gamma^5)u(\nu_\mu)\right] \times \left[\bar{v}(\bar{\nu}_e)\gamma_\beta(1+\gamma^5)u(e^-)\right], \tag{S.15}$$

where $(1-\gamma^5)$ factors become $(1+\gamma^5)$ because $\bar{\gamma}^5 \equiv \gamma^0(\gamma^5)^\dagger\gamma^0 = -\gamma^5$. Consequently,

$$\begin{aligned}|\mathcal{M}|^2 \;=\; \tfrac{1}{2}G_F^2\,&\left[\bar{u}(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\bar{u}(\mu^-)\gamma^\beta(1+\gamma^5)u(\nu_\mu)\right] \\ &\times\left[\bar{u}(e^-)(1-\gamma^5)\gamma_\alpha v(\bar{\nu}_e)\bar{v}(\bar{\nu}_e)\gamma_\beta(1+\gamma^5)u(e^-)\right]\end{aligned} \tag{S.16}$$

and hence

$$\begin{aligned}\tfrac{1}{2}\sum_{\substack{\text{all}\\\text{spins}}}|\mathcal{M}|^2 \;=\; \tfrac{1}{4}G_F^2\,&\mathrm{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu + M_\mu)\gamma^\beta(1+\gamma^5)(\not{p}_\nu + m_\nu)\Big) \\ &\times\mathrm{tr}\Big((1-\gamma^5)\gamma_\alpha(\not{p}_{\bar{\nu}} - m_{\bar{\nu}})\gamma_\beta(1+\gamma^5)(\not{p}_e + m_e)\Big).\end{aligned} \tag{S.17}$$

Please note that here and henceforth the indices μ, e $\nu \equiv \nu_\mu$, and $\bar{\nu} \equiv \bar{\nu}_e$ denote the particles to which respective momenta belong and have nothing to do with the Lorentz indices of those momenta. For the Lorentz indices, I use here α, β and later also γ, δ, σ and ρ. Thus, $p_{\mu\alpha}$ is the α's component of the muon's 4–momentum, *etc.*, *etc.*

Having derived eq. (S.17), we now need to evaluate the traces. For the first trace, we eliminate terms containing odd numbers of γ^ρ matrices and write

$$\begin{aligned}&\mathrm{tr}\left((1-\gamma^5)\gamma^\alpha(\not{p}_\mu + M_\mu)\gamma^\beta(1+\gamma^5)(\not{p}_\nu + m_\nu)\right) \;= \\ &\quad=\;\mathrm{tr}\left((1-\gamma^5)\gamma^\alpha\,\not{p}_\mu\gamma^\beta(1+\gamma^5)\,\not{p}_\nu\right) \;+\; \mathrm{tr}\left((1-\gamma^5)\gamma^\alpha M_\mu\gamma^\beta(1+\gamma^5)m_\nu\right) \\ &\quad=\;\mathrm{tr}\left((1-\gamma^5)\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu(1-\gamma^5)\right) \;+\; M_\mu m_\nu\,\mathrm{tr}\left((1-\gamma^5)\gamma^\alpha\gamma^\beta(1+\gamma^5)\right)\end{aligned}$$

$$\begin{aligned}
&= \operatorname{tr}\Big((1-\gamma^5)^2\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big) \;+\; M_\mu m_e\,\operatorname{tr}\Big((1+\gamma^5)(1-\gamma^5)\gamma^\alpha\gamma^\beta\Big)\\
&= 2\operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big) \;+\; 0\\
&= 2\operatorname{tr}\Big(\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big) \;-\; 2\operatorname{tr}\Big(\gamma^5\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big)\\
&= 8\Big[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] \;+\; 8i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}.
\end{aligned}\tag{S.18}$$

Similarly, the second trace evaluates to

$$\begin{aligned}
&\operatorname{tr}\big((1-\gamma^5)\gamma_\alpha(\not{p}_e+m_e)\gamma_\beta(1+\gamma^5)(\not{p}_{\bar\nu}-m_{\bar\nu}\big) \;=\\
&\qquad\qquad = 8\left[(p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\right] \;+\; 8i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\,.
\end{aligned}\tag{S.19}$$

It remains to substitute the trace formulæ (S.18) and (S.19) back into eq. (S.17) and contract the Lorentz indices. Thus,

$$\begin{aligned}
\tfrac12\sum_{\substack{\text{all}\\ \text{spins}}}|\mathcal{M}|^2 \;&=\; 16G_F^2\left(\Big[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] \,+\, i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\right)\\
&\qquad\times\left(\Big[p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big] +\, i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\right)
\end{aligned}$$

$\langle\!\langle$using symmetry/antisymmetry of factors under $\alpha\leftrightarrow\beta$ $\rangle\!\rangle$

$$\begin{aligned}
&= 16G_F^2\Bigg(\Big[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big]\times\Big[p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big]\\
&\qquad\qquad -\; \epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\times\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\Bigg)\\
&= 16G_F^2\Bigg(\Big[2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}) \,+\, 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e)\\
&\qquad\qquad -\, 2(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu}) \,-\, 2(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu}) \,+\, 4(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})\Big]\\
&\qquad +\; \Big[2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \,-\, 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})\Big]\Bigg)\\
&= 64G_F^2\,(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e)\,.
\end{aligned}\tag{S.20}$$

$\mathcal{Q.E.D.}$

Problem **3**(b):

As explained in the *Peskin & Schroeder* textbook, the partial rate of a decay process (in the rest frame of the initial particle) is given by

$$d\Gamma\;=\;\frac{1}{2M_0}\times\overline{|\mathcal{M}|^2}\times d\mathcal{P} \tag{S.21}$$

where $\mathcal{M}$ is the decay's amplitude, $\overline{|\mathcal{M}|^2}$ is $|\mathcal{M}|^2$ averaged over the unknown initial spins and summed over the unmeasured final spins, and $d\mathcal{P}$ is the infinitesimal phase space factor for the final particles. For three final particles,

$$d\mathcal{P}\;=\;\frac{d^3\mathbf{p}_1}{(2\pi)^3(2E_1)}\,\frac{d^3\mathbf{p}_2}{(2\pi)^3(2E_2)}\,\frac{d^3\mathbf{p}_3}{(2\pi)^3(2E_3)}\times(2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\times(2\pi)\delta(E_1+E_2+E_3-M_0) \tag{S.22}$$

where the energy-momentum conservation law apply in the rest frame, thus $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=\mathbf{p}_{\text{tot}}=\mathbf{0}$ and $E_1+E_2+E_3=E_{\text{tot}}=M_0$.

We start by using the momentum-conservation δ–function to eliminate eliminate the $\mathbf{p}_3$ as independent variable, thus

$$d\mathcal{P}\;=\;\frac{d^3\mathbf{p}_1\,d^3\mathbf{p}_2}{256\pi^5}\;\frac{\delta(E_1+E_2+E_3-E_{\text{tot}})}{E_1E_2E_3}\bigg|_{\mathbf{p}_3=-(\mathbf{p}_1+\mathbf{p}_2)}\,. \tag{S.23}$$

Next, we use spherical coordinates for the two remaining momenta,

$$d^3\mathbf{p}_1\;=\;p_1^2\,dp_1\,d^2\Omega_1\,,\qquad d^3\mathbf{p}_2\;=\;p_2^2\,dp_2\,d^2\Omega_2\,, \tag{S.24}$$

and then replace the $d^2\Omega_2$ describing the direction of the second particle's momentum relative to the fixed external frame with

$$d^2\Omega_2^{(1)}\;=\;d\theta_{12}\,\sin\theta_{12}\,d\phi_2^{(1)}$$

describing the same direction of $\mathbf{p}_2$ relative to the frame centered on the $\mathbf{p}_1$. Consequently,

$$d^2\Omega_1\,d^2\Omega_2\;=\;d^2\Omega_1\,d^2\Omega_2^{(1)}\;=\;\left[d^2\Omega_1\,d\phi_2^{(1)}\right]\,d\theta_{12}\,\sin\theta_{12}\;\equiv\;d^3\Omega\times d(\cos\theta_{12}) \tag{S.25}$$

and hence

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5} \times \frac{p_1^2 p_2^2}{E_1 E_2 E_3}\, dp_1\, dp_2\, d(\cos\theta_{12})\, \delta(E_1 + E_2 + E_3 - E_{\rm tot})\Big|_{\mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2)} . \tag{S.26}$$

Next, we use the cosine theorem

$$p_3^2 \;=\; (\mathbf{p}_1 + \mathbf{p}_2)^2 \;=\; p_1^2 \,+\, p_2^2 \,+\, 2p_1 p_2 \cos\theta_{12}$$

which gives

$$d(\cos\theta_{12}) \;=\; \frac{p_3\, dp_3}{p_1\, p_2}$$

(for fixed p_1, p_2) and therefore

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5} \times \frac{p_1 p_2 p_3}{E_1 E_2 E_3} \times dp_1\, dp_2\, dp_3 \times \delta(E_1 + E_2 + E_3 - E_{\rm tot}). \tag{S.27}$$

Finally, we notice that for a relativistic particle of any mass $pdp = EdE$, hence

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5} \times dE_1\, dE_2\, dE_3\, \delta(E_1 + E_2 + E_3 - E_{\rm tot}), \tag{S.28}$$

and therefore eq. (S.21) for the partial decay rate.

It remains to determine the limits of kinematically allowed ways to distribute the net energy $E_{\rm tot} = M_0$ of the process among the three final particles. Such limits follow from the triangle inequalities for the three momenta,

$$p_1 \;\le\; p_2 \;+\; p_3\,, \qquad p_2 \;\le\; p_1 \;+\; p_3\,, \qquad p_3 \;\le\; p_1 \;+\; p_1\,, \tag{S.29}$$

which look simple but produce rather complicated inequalities for the energies $E_1 = \sqrt{p_1^2 + m_1^2}$, $E_2 = \sqrt{p_2^2 + m_2^2}$, and $E_3 = \sqrt{p_3^2 + m_1^3}$. However, when all three final particles are massless, the kinematic restrictions become simply

$$E_1 \;\le E_2 \;+\; E_3 \;=\; M_0 \;-\; E_1 \tag{S.30}$$

and ditto for the other two inequalities, or equivalently

$$0 \;\le\; E_1, E_2, E_3 \;\le\; \tfrac{1}{2} M_0\,, \qquad \text{while} \quad E_1 + E_2 + E_3 \;=\; M_0\,. \tag{5}$$

Problem **3**(c):

In light of eqs. (3) and (S.21), the partial decay rate of the muon at rest is given by

$$d\Gamma(\mu^- \to e^-\nu_\mu\bar\nu_e) \;=\; \frac{G_F^2}{8\pi^5 M_\mu}\,(p_\mu\cdot p_{\bar\nu})(p_e\cdot p_\nu)\times dE_e\,dE_\nu\,dE_{\bar\nu}\,d^3\Omega\,\delta(E_e+E_\nu+E_{\bar\nu}-M_\mu). \tag{S.31}$$

Specializing to the muon's frame, we have

$$(p_\mu\cdot p_{\bar\nu}) \;=\; M_\mu E_{\bar\nu} \tag{S.32}$$

while

$$\begin{aligned}
(p_e\cdot p_e) \;&=\; E_eE_\nu \;-\; p_ep_\nu\cos\theta_{e\nu}\\
&=\; E_eE_\nu \;+\; \tfrac12 p_e^2 \;+\; \tfrac12 p_\nu^2 \;-\; \tfrac12 p_{\bar\nu}^2\\
&\langle\!\langle\text{neglecting } m_e, m_\nu, m_{\bar\nu}\rangle\!\rangle\\
&=\; E_eE_\nu \;+\; \tfrac12 E_e^2 \;+\; \tfrac12 E_\nu^2 \;-\; \tfrac12 E_{\bar\nu}^2\\
&=\; \tfrac12(E_e+E_\nu)^2 \;-\; \tfrac12 E_{\bar\nu}^2 \;\langle\!\langle\text{using } E_e+E_\nu = M_\mu - E_{\bar\nu}\rangle\!\rangle\\
&=\; \tfrac12 M_\mu(M_\mu - 2E_{\bar\nu}),
\end{aligned} \tag{S.33}$$

Hence,

$$d\Gamma(\mu^- \to e^-\nu_\mu\bar\nu_e) \;=\; \frac{G_F^2}{16\pi^5}\,M_\mu E_{\bar\nu}(M_\mu-2E_{\bar\nu})\times dE_e\,dE_\nu\,dE_{\bar\nu}\,d^3\Omega\,\delta(E_e+E_\nu+E_{\bar\nu}-M_\mu). \tag{S.34}$$

At this point we are ready to integrate over the final-state variables. In light of $\int d^3\Omega = 8\pi^2$ and the kinematic limits (5), we immediately obtain

$$\begin{aligned}
\Gamma(\mu^- \to e^-\nu_\mu\bar\nu_e) \;&=\; \frac{G_F^2 M_\mu}{2\pi^3}\iiint_0^{\frac12 M_\mu} dE_e\,dE_{\bar\nu}\,dE_\nu\,E_{\bar\nu}(M_\mu-2E_{\bar\nu})\delta(E_e+E_\nu+E_{\bar\nu}-M_\mu)\\
&=\; \frac{G_F^2 M_\mu}{2\pi^3}\int_0^{\frac12 M_\mu} dE_e \int_{\frac12 M_\mu - E_e}^{\frac12 M_\mu} dE_{\bar\nu}\,E_{\bar\nu}(M_\mu-2E_{\bar\nu})\\
&=\; \frac{G_F^2 M_\mu}{2\pi^3}\int_0^{\frac12 M_\mu} dE_e\,E_e^2(\tfrac12 M_\mu - \tfrac23 E_e).
\end{aligned} \tag{S.35}$$

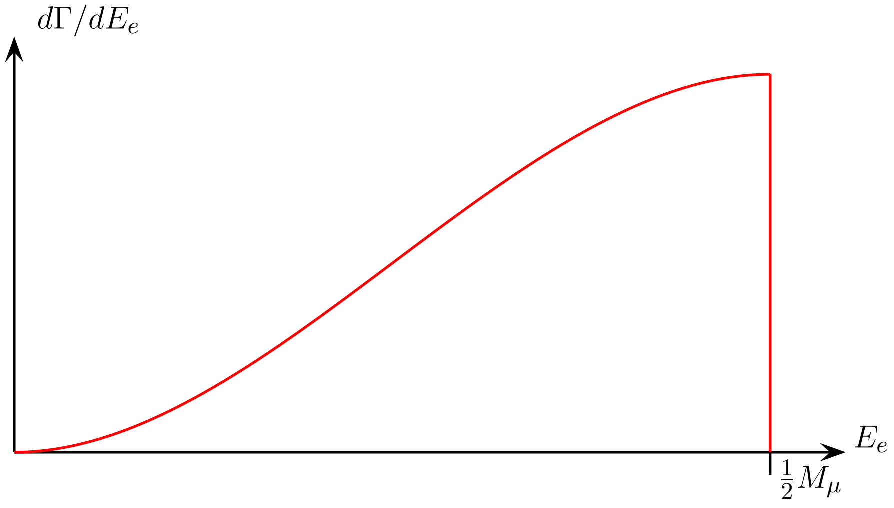
In other words, the partial muon decay rate with respect to the final electron's energy is given by

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2 M_\mu}{12\pi^3} \times E_e^2(3M_\mu - 4E_e) \tag{S.36}$$

or rather

$$\frac{d\Gamma}{dE_e} \approx \begin{cases} \frac{G_F^2}{12\pi^3} M_\mu E_e^2(3M_\mu - 4E_e) & \text{for } E_e < \frac{1}{2}M_\mu, \\ 0 & \text{for } E_e > \frac{1}{2}M_\mu. \end{cases} \tag{S.37}$$

Graphically,



Note how this curve smoothly reaches its maximum at $E_e = \frac{1}{2}M_\mu$ and then abruptly falls down to zero.

It remains to calculate the total decay rate of the muon by integrating the partial rate (S.37) over the electron's energy. The result is

$$\Gamma_{\text{tot}}(\mu \to e\nu\bar{\nu}) = \frac{G_F^2 M_\mu^5}{192\pi^3}\,. \tag{S.38}$$