

Problem 3(a):

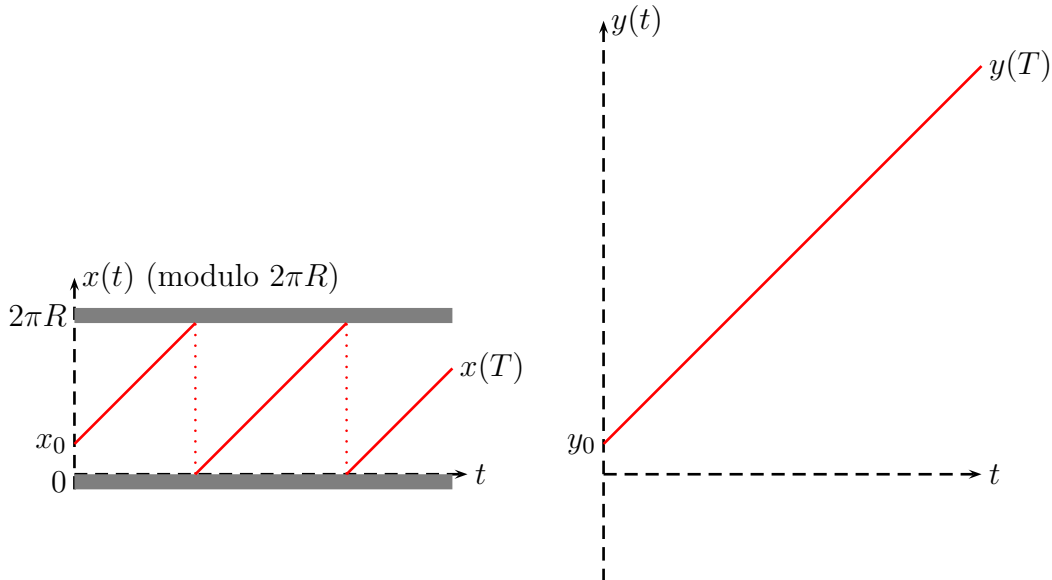
The difference between a circle and a straight line is that on a circle the path of a particle going from point x_0 to point x' does not need to be ‘straight’ but may wrap around the whole circle one or more times. Indeed, let us compare a particle moving on a circle according to $x(t)$ (modulo $2\pi R$) with a particle moving on an infinite line according to $y(t)$. If the two particles have exactly the same velocities at all times,

$$\frac{dx}{dt} \equiv \frac{dy}{dt} \tag{S.1}$$

and similar initial positions $x_0 = y_0$ (according to some coordinate systems) at time $t = 0$, then after time T one generally has

$$y(T) = x(T) + 2\pi R \times n \tag{S.2}$$

for some integer $n = 0, \pm 1, \pm 2, \pm 3, \dots$ because the $x(t)$ path may wrap around the circle n times while the $y(t)$ path may not wrap. For example, the two paths depicted below have same (constant) velocities and begin at $y_0 = x_0$ but end at $y(T) = x(T) + 2\pi R \times 2$:



It is easy to see that the paths $x(t)$ (modulo $2\pi R$) and $y(t)$ (modulo nothing) are in one-to-one correspondence with each other, provided we restrict the initial point y_0 of the particle on

the infinite line to a particular interval of length $L = 2\pi R$, say $0 \leq y_0 < 2\pi R$. Consequently, in the path integral for the particle on the circle

$$\int_{x(t=0)=x_0 \pmod L}^{x(t=T)=x' \pmod L} \mathcal{D}'[x(t) \pmod L] = \sum_{n=-\infty}^{+\infty} \int_{y(t=0)=x_0}^{y(t=T)=x'+nL} \mathcal{D}'[y(t)]. \quad (\text{S.3})$$

Furthermore, in the absence of potential energy, the circle path $x(t) \pmod L$ and the corresponding ∞ line path $y(t)$ have equal actions

$$S[x(t) \pmod L] = S[y(t)] = \int_0^T dt \left[\frac{M}{2} \dot{x}^2 = \frac{M}{2} \dot{y}^2 \right], \quad (\text{S.4})$$

and therefore

$$\begin{aligned} U_{\text{circle}}(x'; x_0) &= \int_{x(t=0)=x_0 \pmod L}^{x(t=T)=x' \pmod L} \mathcal{D}'[x(t) \pmod L] e^{iS[x(t) \pmod L]/\hbar} \\ &= \sum_{n=-\infty}^{+\infty} \int_{y(t=0)=x_0}^{y(t=T)=x'+nL} \mathcal{D}'[y(t)] e^{iS[y(t)]/\hbar} \\ &= \sum_{n=-\infty}^{+\infty} U_{\infty \text{ line}}(y' = x' + nL; y_0 = x_0). \end{aligned} \quad (1)$$

Q.E.D.

Problem 3(b):

For a free particle living on an infinite line the evolution kernel is given by

$$U_{\infty \text{ line}}(y'; y_0) = \sqrt{\frac{M}{2\pi i \hbar T}} \times \exp\left(\frac{i}{\hbar} S_{\text{classical}} = \frac{i}{\hbar} \frac{M(x' - x_0)^2}{2T}\right), \quad (3)$$

hence according to eq. (1), a particle on a circle has kernel

$$U_{\text{circle}}(x'; x_0) = \sqrt{\frac{M}{2\pi i \hbar T}} \times \sum_{n=-\infty}^{+\infty} \exp\left(\frac{iM}{2\hbar T} (x' - x_0 + nL)^2\right). \quad (\text{S.5})$$

To evaluate this sum, we use Poisson re-summation formula (2), which gives

$$\sum_{n=-\infty}^{+\infty} \exp\left(\frac{iM}{2\hbar T}(x' - x_0 + nL)^2\right) = \sum_{\ell=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu \exp\left(\frac{iM}{2\hbar T}(x' - x_0 + \nu L)^2\right) \times e^{2\pi i \ell \nu}. \quad (\text{S.6})$$

Rearranging the exponential, we have

$$\frac{iM}{2\hbar T}(x' - x_0 + \nu L)^2 + 2\pi i \ell \nu = \frac{iML^2}{2\hbar T} \left(\nu + \frac{x' - x_0}{L} + \frac{2\pi \ell \hbar T}{ML^2} \right)^2 - 2\pi i \ell \frac{x' - x_0}{L} - \frac{i\hbar T(2\pi \ell)^2}{ML^2}, \quad (\text{S.7})$$

and therefore

$$\int_{-\infty}^{+\infty} d\nu \exp\left(\frac{iM}{2\hbar T}(x' - x_0 + \nu L)^2\right) \times e^{2\pi i \ell \nu} = \sqrt{\frac{2\pi i \hbar T}{ML^2}} \times \exp\left(-2\pi i \ell \frac{x' - x_0}{L} - \frac{(2\pi \ell)^2 i \hbar T}{ML^2}\right). \quad (\text{S.8})$$

Consequently,

$$\begin{aligned} U_{\text{circle}}(x'; x_0) &= \sqrt{\frac{M}{2\pi i \hbar T}} \times \sqrt{\frac{2\pi i \hbar T}{ML^2}} \times \sum_{\ell=-\infty}^{+\infty} \exp\left(-2\pi i \ell \frac{x' - x_0}{L} - \frac{(2\pi \ell)^2 i \hbar T}{ML^2}\right) \\ &= \frac{1}{L} \sum_{\ell=-\infty}^{+\infty} e^{ip(x' - x_0)/\hbar} \times e^{-iTE/\hbar} \end{aligned} \quad (\text{S.9})$$

where

$$p = -\frac{2\pi \hbar \ell}{L} = -\frac{\hbar \ell}{R} \quad \text{and} \quad E = \frac{p^2}{2M}. \quad (\text{S.10})$$

Problem 3(c): This is obvious from eqs. (S.9) and (S.10).