Problem **9.2**(a):

The Euclidean Lagrangian of scalar QED

$$\mathcal{L}_E = \tfrac{1}{4}F_{\mu\nu}^2 + |\partial_\mu\phi + ieA_\mu\phi|^2 \tag{S.1}$$

comprises the quadratic free part

$$\mathcal{L}_E^{\text{free}} = \tfrac{1}{4}F_{\mu\nu}^2 + |\partial_\mu\phi|^2 \tag{S.2}$$

and the cubic+quartic interaction part

$$\mathcal{L}_E^{\text{int}} = -ieA_\mu\left(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right) + e^2A_\mu^2\phi^*\phi. \tag{S.3}$$

The free part determines the Feynman propagators of the perturbation theory. As discussed in class, the $\frac{1}{4}F_{\mu\nu}^2$ term gives rise to the photon propagator: After suitable gauge fixing, we have

$$A_\mu \sim\!\sim\!\sim\!\sim A_\nu = \frac{1}{p^2}\left(\delta^{\mu\nu} + (\xi-1)\frac{p^\mu p^\nu}{p^2}\right) \tag{S.4}$$

in Euclidean space, and hence

$$A_\mu \sim\!\sim\!\sim\!\sim A_\nu = \frac{i}{p^2+i\epsilon}\left(-g^{\mu\nu} - (\xi-1)\frac{p^\mu p^\nu}{p^2+i\epsilon}\right) \tag{S.5}$$

in Minkowski space.

Next, consider the propagator of the *complex* scalar field $\Phi(x)$. A Gaussian integral over N complex numbers $(z_1,\ldots,z_N)$ *and* their conjugates $(z_1^*,\ldots,z_N^*)$ evaluates to

$$\int\! d^N z \int\! d^N z^* \exp\left(-\sum_{ij} z_i^* a_{ij} z_j\right) = \frac{(2\pi)^N}{\det(A)}\,. \tag{S.6}$$

Note the absence of $\sqrt{}$ on the right hand side of this formula. Likewise, the functional integral over a free complex field *and* its conjugate $\phi^*(x)$ evaluates to

$$\iint\mathcal{D}[\phi(x)]\iint\mathcal{D}[\phi^*(x)]\,\exp\left[-\int d^4x_E\,\phi^*(m^2-\partial^2)\phi\right]\;=\;\frac{\text{const}}{\text{Det}[m^2-\partial^2]}\,. \tag{S.7}$$

Again, the functional determinant $\text{Det}[m^2-\partial^2]$ appears with power -1 rather than $-\frac{1}{2}$. This is different from the functional integral over the real scalar field, because one complex scalar is equivalent to two real scalars.

Now, let's add the source $\eta(x)$. This source is complex, just like the scalar field itself, and it couples to the field according to $2\,\text{Re}(\eta^*\phi)=\eta^*\phi+\eta\phi^*$. In other words, in presence of the source, the free Euclidean action becomes

$$\begin{aligned} S_E^{\text{free}}[\phi(x),\eta(x)] &= \int\left[\phi^*(m^2-\partial^2)\phi\;-\;\eta^*\phi\;-\;\eta\phi^*\right] \\ &= \int\phi'^*((m^2-\partial^2)\phi'\;-\int\eta^*\frac{1}{m^2-\partial^2}\eta, \end{aligned} \tag{S.8}$$

where $\phi'(x)=\phi(x)-(m^2-\partial^2)^{-1}\eta(x)$. As in the real case discussed in class, for the fixed source $\eta(x)$, shifting the scalar field from $\phi(x)$ to $\phi'(x)$ does not affect the measure $\mathcal{D}[\phi(x)]\mathcal{D}[\phi^*(x)]$ of the functional integral, hence

$$\begin{aligned} \mathbf{Z}^{\text{free}}[\eta(x),\eta^*(x)] &= \iint\mathcal{D}[\phi(x)]\iint\mathcal{D}[\phi^*(x)]\,e^{-S_E^{\text{free}}[\phi,\phi^*\eta,\eta^*]} \\ &= \frac{\text{const}}{\text{Det}[m^2-\partial^2]}\times\exp\left[-\int\eta^*\frac{1}{m^2-\partial^2}\eta\right], \end{aligned} \tag{S.9}$$

and therefore

$$\log\mathbf{Z}^{\text{free}}[\eta(x),\eta^*(x)]\;=\;\text{const}\;-\int d^4x_E\int d^4y_E\;\eta^*(x)G_F(x-y)\eta(y) \tag{S.10}$$

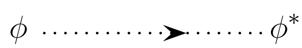
where

$$G_F(x-y)\;=\;\int\frac{d^4p_E}{(2\pi)^4}\,\frac{e^{i(x-y)}}{p_E^2+m^2} \tag{S.11}$$

is the usual scalar propagator, same as for the real scalar field. Consequently, taking variational derivatives of eq. (S.10) with respect to the sources, we have

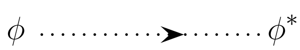
$$\frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta\eta(y)}\log Z \;=\; 0, \qquad \frac{\delta}{\delta\eta^*(x)}\frac{\delta}{\delta\eta^*(y)}\log Z \;=\; 0, \qquad \frac{\delta}{\delta\eta^*(x)}\frac{\delta}{\delta\eta(y)}\log Z \;=\; G_F(x-y). \tag{S.12}$$

Diagrammatically, this means that there are no ϕ–to–ϕ or ϕ^*–to–ϕ^* propagators but only the ϕ–to–ϕ^* or ϕ^*–to–ϕ propagator. Hence, propagators of complex scalars carry arrows to indicate the direction from ϕ to ϕ^*. In momentum basis,



$$= \quad \frac{1}{p^2+m^2} \tag{S.13}$$

in Euclidean space, and hence



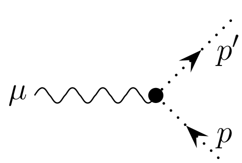
$$= \quad \frac{i}{p^2-m^2+i\epsilon} \tag{S.14}$$

in Minkowski space.

Now consider the vertices of the scalar QED. The interaction Lagrangian (S.3) has a cubic part and a quartic part, which gives us two distinct vertex types, of respected valences 3 and 4. The cubic part contains derivatives of the scalar fields, so to figure out the vertex in momentum basis, we Fourier transform

$$\begin{aligned}
S_E^{\text{cubic}} &= \int d^4x\, ieA_\mu(x)\Big(\phi^*(x)\partial_\mu\phi(x) \;-\; \phi(x)\partial_\mu\phi^*(x)\Big)\\
&= \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\,\Big(\phi^*(-p')\times ip_\mu\phi(p) \;-\; \phi(p)\times(-ip'_\mu)\phi^*(p')\Big)\times\Big(ieA_\mu(p'-p)\Big)\\
&= \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\,\phi(p)\phi^*(-p')A_\mu(p'-p)\times e(p+p')_\mu\,.
\end{aligned} \tag{S.15}$$

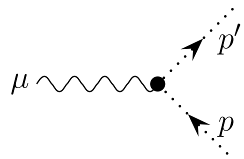
The last factor here gives us the $\phi\phi^*A_\mu$ triple vertex: In Euclidean space



$$= \quad -e(p+p')_\mu \tag{S.16}$$

(where the overall $-$ sign comes from expanding $e^{-S_E^{\text{int}}}$ into a power series), and hence in

Minkowski space

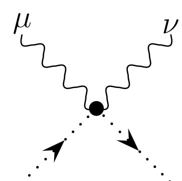


$$= \quad -ie(p+p')_\mu\,. \tag{S.17}$$

Finally, the quartic interaction part

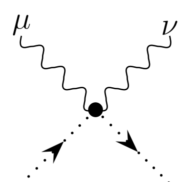
$$L_E^{\text{quartic}} \;=\; e^2 A_\mu A_\mu \phi^* \phi \tag{S.18}$$

of the Euclidean Lagrangian contains no derivatives, so we can derive the quartic “seagull” vertex without the bother of a Fourier transform. In Euclidean space,



$$= \quad -2e^2\delta^{\mu\nu}\,, \tag{S.19}$$

where the factor of 2 comes from combinatorics of two similar photonic lines, and hence in Minkowski space



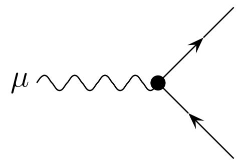
$$= \quad +2ie^2 g^{\mu\nu}\,. \tag{S.20}$$

Problem **9.1**(b):

Now consider the EM field $A_\mu(x)$ coupled both to the electrons' Dirac field $\Psi(x)$ and to a charged scalar field $\phi(x)$,

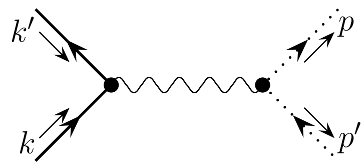
$$\mathcal{L}_E \;=\; \tfrac{1}{4}F_{\mu\nu}^2 \;+\; \overline{\Psi}(\not{D} + m_e)\Psi \;+\; |D_\mu\phi|^2 \;+\; M^2|\phi|^2. \tag{S.21}$$

The Feynman rules of this theory combine those of the ordinary QED and the scalar QED: Photonic, scalar, and fermionic propagators and external lines, and three vertex types shown in eqs. (S.17), (S.20), and (S.22) below:



$$= \quad +ie\gamma^{\mu}. \tag{S.22}$$

At the tree level, there is only one diagram contributing to the $e^- + e^+ \to \phi^+\phi^-$ process, namely



(S.23)

Evaluating this diagram according to Minkowski-space Feynman rules and using Feynman gauge $\xi = 1$, we have

$$i\mathcal{M} \;=\; \bar{v}(k')(ie\gamma^{\mu})u(k) \times -ie(p-p')^{\nu} \times \frac{-ig_{\mu\nu}}{(k+k')^2} \tag{S.24}$$

where the p' momentum appears in the scalar vertex with the $-$ sign because it's an outgoing momentum on an incoming scalar line. (Mismatch between the charge arrow and the momentum direction.) Altogether,

$$\mathcal{M} \;=\; -\frac{e^2}{s}\,(p-p')^{\mu}\,\bar{v}\gamma_{\mu}u\,, \tag{S.25}$$

and for the un-polarized initial particles we need to average $|\mathcal{M}|^2$ over the fermions' spins. Following the usual rules of spin sums, we have

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \;=\; \frac{e^4}{s^2}\,(p-p')^{\mu}(p-p')^{\nu} \times \tfrac{1}{4}\,\mathrm{tr}\Big((\not{k}+m_e)\gamma_{\mu}(\not{k}'-m_e)\gamma_{\nu}\Big) \tag{S.26}$$

where the trace over Dirac indices evaluates to

$$\tfrac{1}{4}\,\mathrm{tr}\Big((\not{k}+m_e)\gamma_{\mu}(\not{k}'-m_e)\gamma_{\nu}\Big) \;=\; k_{\mu}k'_{\nu} \,+\, k_{\nu}k'_{\mu} \,+\, g_{\mu\nu}(m_e^2 - kk'). \tag{S.27}$$

Hence, neglecting the electron's mass, we arrive at

$$|\mathcal{M}|^2_{\text{avg}} \;=\; \frac{e^4}{s^2} \times \big[2(kp-kp')(k'p-k'p') \;-\; (kk')(p-p')^2\big] \tag{S.28}$$

The rest is kinematics. In the center-of-mass frame we have

$$\begin{aligned}
s \;&=\; (k+k')^2 \;=\; 4E^2,\\
kp \;=\; k'p' \;&=\; E^2 \;-\; |\mathbf{k}||\mathbf{p}|\cos\theta \;\approx\; E^2(1-\beta\cos\theta),\\
k'p \;=\; kp' \;&=\; E^2 \;+\; |\mathbf{k}||\mathbf{p}|\cos\theta \;\approx\; E^2(1+\beta\cos\theta),\\
kk' \;&=E^2 \;+\; |\mathbf{k}|^2 \;\approx\; 2E^2,\\
(p-p')^2 \;&=\; -|2\mathbf{p}|^2 \;=\; -4E^2\beta^2,
\end{aligned} \tag{S.29}$$

where

$$\beta \;=\; \frac{|\mathbf{p}|}{E} \;=\; \sqrt{1-\frac{M^2}{E^2}} \tag{S.30}$$

is the velocity of (each) final scalar particle, and the approximation is neglecting the electrons mass (but not the scalar mass). Consequently

$$2(kp-kp')(k'p-k'p') \;-\; (kk')(p-p')^2 \;=\; -8E^4\beta^2\cos^2\theta \;+\; 8E^4\beta^2 \;=\; +8E^4\beta^2\sin^2\theta, \tag{S.31}$$

hence

$$|\mathcal{M}|^2_{\text{avg}} \;=\; \tfrac{1}{2}e^4\beta^2\sin^2\theta \tag{S.32}$$

and therefore the partial pair-production cross-section is

$$\frac{d\sigma}{d\Omega_{\text{cm}}} \;=\; \frac{|\mathbf{p}|}{|\mathbf{k}|}\,\frac{1}{64\pi^2 s}\;|\mathcal{M}|^2_{\text{avg}} \;=\; \frac{\alpha^2}{8s}\,\beta^3\,\sin^2\theta, \tag{S.33}$$

and the total cross-section is

$$\sigma \;=\; \frac{\pi\alpha^2}{3s}\times\beta^3. \tag{S.34}$$

Note the angular dependence of the scalar pair production is very different from the fermionic pair productions: Unlike the $\mu^-\mu^+$ or $q\bar{q}$ pairs which favor the directions along the beam axis, the scalar pairs would rather fly off sideways. The total cross-section is also quite different: Four times smaller at energies well above the threshold (($\pi\alpha^2/3s$ for scalar pairs versus $4\pi\alpha^2/3s$ for muon pairs), and even smaller near the threshold. Indeed, near the threshold, the scalar pair-production rate rises only as β^3, which makes it rather difficult to detect experimentally!

Problems **9.2**(a,b) are explained in a supplementary note (see the web site).
Please read those explanations first.

For ease of reference, let me repeat the rules for calculating the partition function $Z(\beta) \equiv \operatorname{Tr}\left(e^{-\beta\hat{H}}\right)$ of a quantum particle for a (real) temperature $\mathcal{T} = 1/\beta$: Z is given by the *Euclidean path integral* subject to periodic boundary conditions for particles position $x(t_E)$ with respect to the *Euclidean time* t_E: $x(t_E = \beta) = x(t_E = 0)$. Thus,

$$Z(\mathcal{T}) \;=\; \int\limits^{x(\beta)=x(0)} \mathcal{D}[x(t_E)]\, e^{-S_E[x(t_E)]} \tag{S.35}$$

where the Euclidean Action is

$$S_E[x(t_E)] \;=\; \int\limits_0^\beta dt_E \left(\tfrac{1}{2}\left(\frac{dx}{dt_E}\right)^2 \,+\, V(x)\right). \tag{S.36}$$

Problem **9.2**(c):
Generalizing eq. (S.35) from particle mechanics to field theory is quite straightforward. For a real scalar field field $\phi(x)$ with a Euclidean Lagrangian

$$\mathcal{L}_E \;=\; \tfrac{1}{2}(\partial\phi)^2 \,+\, V(\phi) \tag{S.37}$$

we have finite-temperature Partition function

$$Z(\beta) \;=\; \iint\limits^{\phi(\mathbf{x},\beta)=\phi(\mathbf{x},0)} \mathcal{D}[\phi(\mathbf{x},x_4)]\, \exp\left[-\int d^3\mathbf{x} \int\limits_0^\beta dx_4 \left(\tfrac{1}{2}(\partial\phi)^2 \,+\, V(\phi)\right)\right]. \tag{S.38}$$

In other words, finite temperature translates into geometry of the Euclidean 4D spacetime: The Euclidean time $x_4 = it$ is of finite extent $\beta = 1/\mathcal{T}$ and the scalar field is subject to the periodic boundary condition; the other 3 dimensions x_1, x_2, x_3 are infinite as usual.

For the free scalar field, the Euclidean action is a quadratic functional

$$S_E[\phi(x_E)] \;=\; \tfrac{1}{2}\int d^4x_E\, \phi(m^2-\partial^2)\phi\,, \tag{S.39}$$

which becomes diagonal after a Fourier transform. However, because of the periodicity of the Euclidean time coordinate, the Euclidean "energies" k_4 have discrete rather than continuous spectrum,

$$\phi(x_E) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\frac{1}{\sqrt{\beta}}e^{ik_Ex_E}\Phi(k_E) \tag{S.40}$$

where

$$k_4 \;=\; \frac{2\pi}{\beta}\times\text{integer}. \tag{S.41}$$

Consequently,

$$S_E \;=\; \tfrac{1}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}(m^2+k_E^2)\times|\Phi(k_E)|^2 \tag{S.42}$$

and hence

$$Z \;=\; \left[\operatorname{Det}(m^2-\partial_E^2)_{\text{periodic}}\right]^{-1/2} \;=\; \exp\left[\text{const}-\tfrac{1}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\log(m^2+k^2)\right]. \tag{S.43}$$

It is often convenient to re-express a sum over a discrete momentum component using Poisson's re-summation formula:

$$\begin{aligned}\sum_{n=-\infty}^{+\infty}F(n) \;&=\; \int_{-\infty}^{+\infty}dx\,F(x)\times\sum_{n=-\infty}^{+\infty}\delta(x-n)\\ &=\; \int_{-\infty}^{+\infty}dx\,F(x)\times\sum_{\ell=-\infty}^{+\infty}e^{2\pi i\ell x}\end{aligned} \tag{S.44}$$

or for the problem at hand

$$\sum_{k_4}F(k_4) \;=\; \beta\sum_{\ell=-\infty}^{+\infty}\int\frac{dk_4}{2\pi}\,F(k_4)e^{i\beta\ell k_4}. \tag{S.45}$$

Hence, the Helmholtz free energy $\mathcal{F}=-\mathcal{T}\log Z$ of the free Hermitian scalar field can be written as

$$\mathcal{F} \;=\; \text{const} \;+\; \tfrac{1}{2}\sum_{\ell=-\infty}^{+\infty}\int\!\frac{d^4k_E}{(2\pi)^4}\,e^{i\beta\ell k_4}\,\log(k_E^2+m^2). \tag{S.46}$$

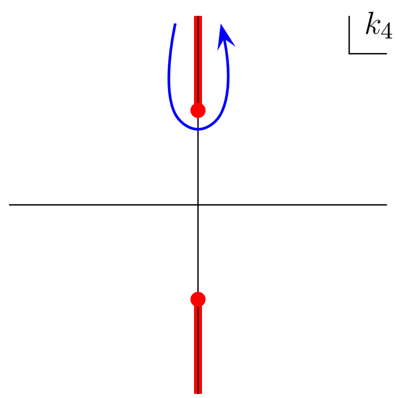
In the zero-temperature limit $\beta\to\infty$, the sum $\sum_\ell$ reduces to the $\ell=0$ term while all the other terms are suppressed by the rapidly changing phase $e^{i\beta\ell k_4}$. In the general spirit of subtracting the zero-point energy contribution, we should therefore get rid of the $\ell=0$ term. Since all the other terms come in symmetric pairs $\pm\ell\neq 0$, we arrive at

$$\mathcal{F} \;=\; \sum_{\ell=1}^{\infty}\int\!\frac{d^4k_E}{(2\pi)^4}\,e^{i\beta\ell k_4}\,\log(k_E^2+m^2). \tag{S.47}$$

Formula (S.47) has a nice 4D form, but for the purpose of comparison with the ordinary statistical mechanics, let us integrate over the k_4 before we integrate over the 3–momentum $\mathbf{k}$. For fixed $\mathbf{k}$ and ℓ we are faced with the integral

$$I \;=\; \int\!\frac{dk_4}{2\pi}\,e^{i\beta\ell k_4}\,\log(k_4^2+E^2) \tag{S.48}$$

where $e^2=m^2+\mathbf{k}^2$. The logarithm here has branch cuts (in the complex k_4 plane) from $+iE$ to $+i\infty$ and also from $-iE$ to $-i\infty$, so we would like to deform the integration contour away from the real axis to make it wrap around the upper branch cut:



In other words, $k_4=iE(1+x+i\epsilon)$ on its way down from $x=+\infty$ to $x=0$ and $k_4=iE(1+x-i\epsilon)$

on its way up from $x=0$ back to $k=+\infty$, hence

$$\begin{aligned} I \;&=\; \frac{iE}{2\pi}\int\limits_0^{+\infty}\! dx\, e^{-\beta\ell E(1+x)} \times \Big[\log(E^2(-2x-x^2+i\epsilon))\;-\;\log(E^2(-2x-x^2-i\epsilon))\;=\;2\pi i\Big] \\ &=\; -E\int\limits_0^{+\infty}\! dx\, e^{-\beta\ell E(1+x)} \;=\; -\frac{e^{-\beta\ell E}}{\beta\ell}\,. \end{aligned} \tag{S.49}$$

Next, we sum this integral over ℓ, which gives gives

$$-\frac{1}{\beta}\sum_{\ell=1}^{\infty}\frac{(e^{-\beta E})^\ell}{\ell} \;=\; \mathcal{T}\log\Big(1\,-\,e^{-\beta E}\Big) \tag{S.50}$$

and therefore free energy

$$\mathcal{F}(\mathcal{T},m) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathcal{T}\log\Big(1\,-\,e^{-\beta E_{\mathbf{k}}}\Big) \tag{S.51}$$

Finally, let us compare our result (S.51) with the conventional statistical mechanics of identical spinless relativistic bosons. In SM of identical bosons,

$$\mathcal{F}(\mathcal{T},m) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathcal{F}^{\text{harmonic}}_{\text{oscillator}}(\mathcal{T},E_{\mathbf{k}}) \tag{S.52}$$

where each oscillator mode contributes

$$\mathcal{F}^{\text{harmonic}}_{\text{oscillator}}(\mathcal{T},E_{\mathbf{k}}) \;=\; -\mathcal{T}\log Z^{\text{harmonic}}_{\text{oscillator}} \;=\; \mathcal{T}\log(2\sinh(E\beta/2)) \;=\; \tfrac{1}{2}E\;+\;\mathcal{T}\log\Big(1\,-\,e^{-\beta E}\Big) \tag{S.53}$$

Subtracting the zero-point energy $\frac{1}{2}E$ and substituting into eq. (S.52) we arrive at precisely eq. (S.51), which shows that the Functional Quantization of the field theory correctly reproduces the free energy of the field's quanta.

Problem **9.2**(d):

For the $(0+1)$ dimensional (zero space, one time) free complex Grassmann field $\psi(t)$ we have quadratic Euclidean action

$$S_E \;=\; \int_0^\beta dt_E\, \bar{\psi}(\partial+\omega)\psi \tag{S.54}$$

and hence Partition Function

$$Z \;=\; \mathrm{Det}[\partial+\omega]\,. \tag{S.55}$$

All physical observables of this system must be periodic in Euclidean time, so the odd Grassmannians such as the fermionic fields themselves should be either periodic or antiperiodic. Therefore, the 'momentum' modes should be quantized as either integers or half integers,

$$k \;=\; \frac{2\pi}{\beta}\times n \qquad \text{or} \quad k \;=\; \frac{2\pi}{\beta}\times(n+\tfrac{1}{2}), \tag{S.56}$$

which produces two distinct expressions for the partition function: In the periodic case, we have

$$\begin{aligned} Z_+ \;&\propto\; \prod_k (ik+\omega) \\ &=\; \omega\times\prod_{n=1}^{\infty}\left(\omega^2+\left(\frac{2\pi n}{\beta}\right)^2\right) \\ &\propto\; \beta\omega\times\prod_{n=1}^{\infty}\left(1\;+\;\left(\frac{\beta\omega}{2\pi n}\right)^2\right) \\ &=\; 2\sinh(\beta\omega/2), \end{aligned} \tag{S.57}$$

while the anti-periodic partition function is

$$\begin{aligned}
Z_- \;&\propto\; \prod_k (ik+\omega) \\
&=\; \prod_{n=0}^{\infty}\left(\omega^2+\left(\frac{2\pi}{\beta}\right)^2(n+\tfrac{1}{2})^2\right) \\
&\propto\; \prod_{n=0}^{\infty}\left(1\,+\,\left(\frac{\beta\omega}{2\pi(n+\frac{1}{2})}\right)^2\right) \\
&=\; \prod_{m=1}^{\infty}\left(1\,+\,\left(\frac{\beta\omega}{2\pi(m/2)}\right)^2\right)\Big/\prod_{m=1}^{\infty}\left(1\,+\,\left(\frac{\beta\omega}{2\pi m}\right)^2\right) \\
&=\; \sinh(\beta\omega)\Big/\sinh(\beta\omega/2) \\
&=\; 2\cosh(\beta\omega/2).
\end{aligned} \tag{S.58}$$

In other words,

$$Z_\pm(\beta,\omega) \;=\; e^{+\beta\omega/2}\left(1\,\mp\,e^{-\beta\omega}\right). \tag{S.59}$$

Physically, the anti-periodic partition function Z_- agrees with the two-level Fermi statistics while the periodic function Z_+ does not seem to be a partition function of anything.

Therefore, *at finite temperature, the fermionic fields are anti-periodic in the Euclidean time.*

Naturally, the same rule applies to the fermionic fields in any space dimension. For example, for a free Dirac field in $d=(3+1)$, we have

$$Z \;=\; \left[\mathrm{Det}(m^2-\partial_E^2)_{\text{antiperiodic}}\right]^{+2} \;=\; \text{const}\times\exp\left[+2\int\frac{d^3\mathbf{x}}{(2\pi)^3}\sum_{k_4}\log(k_E^2+m^2)\right] \tag{S.60}$$

where the k_4 "energies" have the half-integral rather than integral spectrum. Consequently, the Poisson re-summation becomes

$$\sum_{k_4}F(k_4) \;=\; \beta\sum_{\ell=-\infty}^{+\infty}\int\frac{dk_4}{2\pi}\,F(k_4)\times e^{i\beta\ell k_4}\times(-1)^\ell, \tag{S.61}$$

which leads to the free energy

$$\mathcal{F} \;=\; \text{const} \;-\; 2\sum_{\ell=-\infty}^{+\infty}(-1)^\ell\int\!\frac{d^4k_E}{(2\pi)^4}\,e^{i\beta\ell k_4}\,\log(k_E^2+m^2) \tag{S.62}$$

or, after subtracting the zero-point energy

$$\mathcal{F} \;=\; 4\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\int\!\frac{d^4k_E}{(2\pi)^4}\,e^{i\beta\ell k_4}\,\log(k_E^2+m^2) \tag{S.63}$$

Similar to the bosonic case, we may re-write this formula in the conventional 3D terms by integration over the k_4 and summing over the ℓ's. The integration over the k_4 works exactly as in the bosonic case, but the sum $\sum_\ell$ is slightly different because of the alternating signs:

$$\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{\beta\ell}\,e^{-\beta E\ell} \;=\; -\mathcal{T}\,\log\Bigl(1\,+\,e^{-\beta E}\Bigr) \tag{S.64}$$

and hence

$$\mathcal{F}(\mathcal{T},m) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,(-4\mathcal{T})\,\log\Bigl(1\,+\,e^{-\beta E_{\mathbf{k}}}\Bigr) \tag{S.65}$$

in full agreement with the Fermi–Dirac statistical mechanics.

Problem **9.2**(e):

Similarly to other bosonic fields, at finite temperature $\mathcal{T}=1/\beta$, the EM field $A^\mu(x_E)$ becomes periodic in the Euclidean time direction,

$$A^\mu(\mathbf{x},x_4=0) \;=\; A^\mu(\mathbf{x},x_4=\beta), \qquad \mu=1,2,3,4. \tag{S.66}$$

Its *local* properties however remain exactly the same; in particular, we still have local gauge transformations

$$A'^\mu(x_E) \;=\; A^\mu(x_E) \;-\; \partial^\mu\Lambda(x_E) \tag{S.67}$$

albeit subject to the periodicity condition

$$\partial^\mu\Lambda(\mathbf{x},x_4=0) \;=\; \partial^\mu\Lambda(\mathbf{x},x_4=\beta). \tag{S.68}$$

Consequently, the proper construction of the Euclidean Functional integral over the EM field configurations requires the same Fadde'ev–Popov gauge-fixing procedure as for $\mathcal{T}=0$ with suitable modifications to reflect the fields' periodicities. Thus, the EM Partition Function is

$$Z_{\mathrm{EM}} \;=\; C \int\limits_{\text{periodic}} \mathcal{D}[A^\mu(x_E)]\,\Delta_{\mathrm{FP}} e^{-S_E[A^\mu(x_E)]} \tag{S.69}$$

where the Euclidean action

$$S_E[A^\mu(x_E)] \;=\; \int d^3\mathbf{x} \int\limits_0^\beta dx_4\,\left\{\tfrac{1}{4}F^2_{\mu\nu} \;+\; \tfrac{1}{2\xi}(\partial A)^2\right\} \tag{S.70}$$

includes the gauge-fixing term and the Fadde'ev–Popov determinant

$$\Delta_{\mathrm{FP}} \;=\; \mathrm{Det}(-\partial^2)_{\text{periodic}} \tag{S.71}$$

takes into account the periodicity (S.68) of the finite-temperature gauge transformations. Finally, the normalization factor

$$C \;=\; \left[\iint \mathcal{D}[\omega(x_E)]\,e^{-\frac{1}{2\xi}\int \omega^2 d^4x_E}\right]^{-1} \tag{S.72}$$

compensating for the averaging over the gauge conditions $\partial_\mu A^\mu = \omega$ should also involve properly periodic $\omega(x_E)$.

For the free EM field, the Euclidean action functional (S.70) is quadratic and the functional integral (S.69) is purely Gaussian, but keeping in mind the Fadde'ev–Popov determinant factor Δ_{FP}, we have

$$Z_{\mathrm{EM}} \;=\; C\,\mathrm{Det}(-\partial^2) \times \left[\mathrm{Det}\left(-\partial^2\delta^{\mu\nu} + (1-\xi^{-1})\partial^\mu\partial^\nu\right)\right]^{-1/2} \tag{S.73}$$

where all the determinants are over periodic fields. Hence, in the momentum basis

$$\log Z_{\mathrm{EM}} \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\left\{\tfrac{1}{2}\log(\xi^{-1}) \;+\; \log(k_E^2) \;-\; \tfrac{1}{2}\log\det\left(k_E^2\delta^{\mu\nu} - (1-\xi^{-1})k_E^\mu k_E^\nu\right)\right\} \tag{S.74}$$

where the last determinant acts on the Euclidean indices μ,ν only.

The 4×4 matrix $(k_E^2\delta^{\mu\nu}-(1-\xi^{-1})k_E^\mu k_E^\nu)$ has three eigenvalues equal to k_E^2 (transverse eigenvectors) and one eigenvalue equal to k_E^2/ξ (eigenvector parallel to the k_E). Consequently

$$\det\left(k_E^2\delta^{\mu\nu}-(1-\xi^{-1})k_E^\mu k_E^\nu\right)\ =\ \frac{(k_E^2)^4}{\xi}\,, \tag{S.75}$$

and therefore

$$\tfrac{1}{2}\log(\xi^{-1})\ +\ \log(k_E^2)\ -\ \tfrac{1}{2}\log\det\left(k_E^2\delta^{\mu\nu}-(1-\xi^{-1})k_E^\mu k_E^\nu\right)\ =\ -\log(k_E^2). \tag{S.76}$$

Plugging this formula into eq. (S.74) finally gives us the electromagnetic partition function,

$$\log Z_{\mathrm{EM}}\ =\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\left\{-1\times\log(k_E^2)\right\}. \tag{S.77}$$

By comparison, a (real) scalar field has

$$\log Z_\phi\ =\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\left\{-\tfrac{1}{2}\times\log(k_E^2+m^2)\right\}, \tag{S.78}$$

which means the EM field has the partition function of *two species* of a massless scalar — or equivalently, two physical polarizations states of one massless vector species, the photon. $\mathcal{Q.E.D.}$