

Problem 1:

In the higher-derivative regularization scheme the scalar propagators are UV-softened according to eq. (1), hence the product of the two propagators in the diagram (2) becomes

$$\frac{1}{q_1^2 + m^2} \times \frac{1}{q_2^2 + m^2} - \frac{1}{q_1^2 + \Lambda^2} \times \frac{1}{q_2^2 + m^2} - \frac{1}{q_1^2 + m^2} \times \frac{1}{q_2^2 + \Lambda^2} + \frac{1}{q_1^2 + \Lambda^2} \times \frac{1}{q_2^2 + \Lambda^2} \quad (\text{S.1})$$

where $q_2 = k - q_1$. Applying Feynman's parameter trick to each of these products, we obtain

$$\int_0^1 dx \left(\frac{1}{[q^2 + \Delta]^2} - \frac{1}{[q^2 + \Delta + x\tilde{\Lambda}^2]^2} - \frac{1}{[q^2 + \Delta + (1-x)\tilde{\Lambda}^2]^2} + \frac{1}{[q^2 + \Delta + \tilde{\Lambda}^2]^2} \right) \quad (\text{S.2})$$

where $q = q_1 - kx$ is the same in all four terms,

$$\Delta(x) = m^2 + x(1-x)k_E^2 = m^2 - x(1-x)k_{\text{Mink}}^2 \quad (\text{S.3})$$

is also the same in all the terms, and finally

$$\tilde{\Lambda}^2 = \Lambda^2 - m^2 \approx \Lambda^2 \quad (\text{S.4})$$

is what makes the four terms different from each other.

Now we need to integrate the propagator product over the Euclidean momentum. As in class, we integrate over the momentum before integrating over x in eq. (S.2), and this allows us to shift the integration variable from q_1 (or q_2) to q and use spherical symmetry. Thus,

$$d^4 q_E = 2\pi^2 q^3 dq = \pi^2 q^2 dq^2, \quad (\text{S.5})$$

and therefore

$$\begin{aligned}
& \int \frac{d^4 q_E}{(2\pi)^4} \left(\frac{1}{[q^2 + \Delta]^2} - \frac{1}{[q^2 + \Delta + x\tilde{\Lambda}^2]^2} - \frac{1}{[q^2 + \Delta + (1-x)\tilde{\Lambda}^2]^2} + \frac{1}{[q^2 + \Delta + \tilde{\Lambda}^2]^2} \right) \\
&= \frac{1}{16\pi^2} \int_0^\infty dq^2 \left[\begin{aligned} & \frac{q^2}{[q^2 + \Delta]^2} - \frac{q^2}{[q^2 + \Delta + x\tilde{\Lambda}^2]^2} \\ & - \frac{q^2}{[q^2 + \Delta + (1-x)\tilde{\Lambda}^2]^2} \\ & + \frac{q^2}{[q^2 + \Delta + \tilde{\Lambda}^2]^2} \end{aligned} \right] \\
&= \frac{1}{16\pi^2} \left(\begin{aligned} & \left(\log(q^2 + \Delta) - \frac{q^2}{q^2 + \Delta} \right) \\ & - \left(\log(q^2 + \Delta + x\tilde{\Lambda}^2) - \frac{q^2}{q^2 + \Delta + x\tilde{\Lambda}^2} \right) \\ & - \left(\log(q^2 + \Delta + (1-x)\tilde{\Lambda}^2) - \frac{q^2}{q^2 + \Delta + (1-x)\tilde{\Lambda}^2} \right) \\ & + \left(\log(q^2 + \Delta + \tilde{\Lambda}^2) - \frac{q^2}{q^2 + \Delta + \tilde{\Lambda}^2} \right) \end{aligned} \right) \Bigg|_{q^2=0}^{q^2=\infty} \\
&= \log \frac{(\Delta + x\tilde{\Lambda}^2) \times (\Delta + (1-x)\tilde{\Lambda}^2)}{\Delta \times (\Delta + \tilde{\Lambda}^2)} \\
&\approx \log \frac{x\tilde{\Lambda}^2 \times (1-x)\tilde{\Lambda}^2}{\Delta \times \tilde{\Lambda}^2} \\
&\approx \log \frac{x(1-x)\Lambda^2}{\Delta}.
\end{aligned} \tag{S.6}$$

Consequently, the whole diagram evaluates to

$$\begin{aligned}
\mathcal{M} &= \frac{\lambda^2}{2} \times \frac{1}{16\pi^2} \int_0^1 dx \log \frac{x(1-x)\Lambda^2}{\Delta = m^2 - x(1-x)k^2} \\
&= \frac{\lambda^2}{32\pi^2} \left[\log \frac{\Lambda^2}{m^2} + \int_0^1 dx \log x(1-x) - \int_0^1 dx \log \left(1 - x(1-x)\frac{k^2}{m^2} \right) \right] \\
&= \frac{\lambda^2}{32\pi^2} \left[\log \frac{\Lambda^2}{m^2} - 2 + I(k^2/m^2) \right].
\end{aligned} \tag{S.7}$$

Finally, let us compare our result (S.7) for the higher-derivative UV regulators with

$$\mathcal{M} = \frac{\lambda^2}{32\pi^2} \left[\log \frac{\Lambda^2}{m^2} - 1 + I(k^2/m^2) \right] \tag{S.8}$$

we have obtained in class for the hard-edge UV cutoff. Clearly the only difference between the two formulæ is the numerical constant inside the square brackets. Moreover, this difference may be absorbed into a re-definition of the UV cutoff parameter: If we set

$$\Lambda_{\text{higher-derivative}}^2 = \exp(1) \times \Lambda_{\text{hard-edge}}^2, \tag{S.9}$$

then

$$\log \frac{\Lambda_{HD}^2}{m^2} - 2 + I(k^2/m^2) = \log \frac{\Lambda_{HE}^2}{m^2} - 1 + I(k^2/m^2) \tag{S.10}$$

and eqs. (S.7) and (S.8) are in perfect agreement. *Q.E.D.*