Problem 1:
Our starting point is eq. (3) where \( x + y + z \equiv 1 \) and \( q_3 = p - q_1 - q_2 \). Expanding \( q_3^2 \) and collecting all the terms containing the \( q_1 \) momentum into a full square, we find

\[
xq_1^2 + yq_2^2 + z(q - q_1 - q_2)^2 = (x+z)
\left(q_1 + \frac{z}{x+z} (q_2 - p)\right)^2 + \frac{xz}{x+z} (q_2-p)^2 + yq_2^2. \tag{S.1}
\]

Naturally, we interpret the first term here as \( \alpha k_1^2 \), thus

\[
\alpha = (x+z), \quad k_1 = q_1 + \frac{z}{x+z} (q_2 - p). \tag{S.2}
\]

For the other two terms on the right hand side of eq. (S.1), we expand \( (q_2 - p)^2 \) and collect all terms containing the \( q_2 \) momentum into another full square, thus

\[
\frac{xz}{x+z} (q_2-p)^2 + yq_2^2 = \frac{xz + y(x+z)}{x+z} \left(q_2 - \frac{xz}{xz+y(x+z)} p\right)^2 + \frac{xzy}{xz+y(x+z)} p^2. \tag{S.3}
\]

Consequently, we define

\[
\beta = \frac{xy + xz + yz}{x+z}, \quad \gamma = \frac{xyz}{xy+xz+yz}, \quad k_2 = q_2 - \frac{xz}{xy+xz+yz} p, \tag{S.4}
\]

which makes the right hand side of eq. (S.3) into \( \beta k_2^2 + \gamma p^2 \). Altogether, we have

\[
xq_1^2 + yq_2^2 + zq_3^2 = \alpha k_1^2 + \beta k_2^2 + \gamma p^2 \tag{S.5}
\]

and hence eq. (4).
Finally, we need to check the Jacobian of replacing the original independent loop momenta \( q_1 \) and \( q_2 \) with \( k_1 \) and \( k_2 \). In light of eqs. (S.2) and (S.4), it is easy to see that

\[
\frac{\partial(q_1, q_2)}{\partial(k_1, k_2)} = \det \begin{pmatrix} 1 & \frac{z}{z + z} \\ 0 & 1 \end{pmatrix} = 1, \tag{S.6}
\]

and therefore \( dk_1 \, dk_2 = dq_1 \, dq_2 \), dimension by dimension. In other words, for fixed Feynman parameters,

\[
\int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} = \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4}. \tag{S.7}
\]

**Problem 2:**
Assembling all the factors of the two loop amplitude (1) and making use of eqs. (2), (4), and (S.7), we have

\[
-i \Sigma(p^2) = \frac{(-i\lambda)^2}{3!} \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i0} \frac{i}{q_3^2 - m^2 + i0} \frac{i}{q_3^2 - m^2 + i0} \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{2}{D^3} \tag{S.8}
\]

where \( D \) is as in eq. (4). In particular, the dependence on the external momentum \( p \) comes solely through the \( \gamma p^2 \) term in \( D \), hence

\[
\frac{d\Sigma}{dp^2} = -\frac{\lambda^2}{6} \int \int dx \, dy \, dz \, \delta(x + y + z - 1) \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{-6\gamma}{D^4}. \tag{S.9}
\]

Note that for large loop momenta \( k_1 \) and \( k_2 \), \( D \) grows like \( k^2 \). Consequently, the integrand of the 8–dimensional momentum integral (S.8) behaves like \( 1/k^6 \), so the integral diverges quadratically. On the other hand, the integrand of (S.9) behaves like \( 1/k^8 \), so the divergence of this integral is only logarithmic.
Problem 3:
Rotating both loop momenta \( k_1 \) and \( k_2 \) into Euclidean momentum space, we have \( d^4k_1 \rightarrow id^4k_1^E \); \( d^4k_2 \rightarrow id^4k_2^E \), and

\[
\mathcal{D} \rightarrow -\alpha(k_1^E)^2 - \beta(k_2^E)^2 + \gamma p^2 - m^2, \tag{S.10}
\]

hence

\[
\frac{d\Sigma}{dp^2} = \frac{\lambda^2}{6} \iiint dx \, dy \, dz \, \delta(x + y + z - 1) \int d^4k_1^E \int d^4k_2^E \frac{6 \times (-\gamma)}{[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2]^4}. \tag{S.11}
\]

Next, we need dimensional regularization to actually perform the momentum integrals. Changing

\[
\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \rightarrow \mu^{2(4-D)} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \tag{S.12}
\]

(Euclidean signature for all dimensions), we have

\[
\mu^{8-2D} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \frac{6}{[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2]^4} = \]

\[
\langle \langle \text{using eq. (6)} \rangle \rangle
\]

\[
= \mu^{8-2D} \int_0^\infty dt \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \exp \left( -t[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2] \right)
\]

\[
= \mu^{8-2D} \int_0^\infty dt \, t^3 \, e^{-t(m^2-\gamma p^2)} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \, e^{-t\alpha k_1^2} \, e^{-t\beta k_2^2}
\]

\[
\langle \langle \text{using eq. (7)} \rangle \rangle
\]

\[
= \mu^{8-2D} \int_0^\infty dt \, t^3 \, e^{-t(m^2-\gamma p^2)} \times (4\pi \alpha t)^{-D/2} (4\pi \beta t)^{-D/2}
\]

\[
= \frac{\mu^{8-2D}}{(4\pi)^D (\alpha \beta)^{D/2}} \times \int_0^\infty dt \, t^{3-D} \, e^{-t(m^2-\gamma p^2)}
\]

\[
= \frac{\mu^{8-2D}}{(4\pi)^D (\alpha \beta)^{D/2}} \times \Gamma(4 - D)(m^2 - \gamma p^2)^{D-4}. \tag{S.13}
\]
Note the $\Gamma(4-D)$ factor: It has a pole at $D = 4$ but no poles at $D < 4$. This is dimensional regularization’s way to show that the momentum integrals diverge, but only logarithmically.

At this point, we may take $D = 4 - \varepsilon$ for an infinitesimally small $\varepsilon$. Hence, the last line of eq. (S.13) becomes

$$
\frac{1}{(4\pi)^4(\alpha\beta)^2} \Gamma(2\varepsilon) \left( \frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2 - \gamma p^2} \right)^{2\varepsilon} \xrightarrow{\varepsilon \to 0} \frac{1}{(4\pi)^4(\alpha\beta)^2} \times \left( \frac{1}{2\varepsilon} - \gamma_E + \log \frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2 - \gamma p^2} \right). \tag{S.14}
$$

Plugging this formula back into eq. (S.11) and assembling all the factors, we finally arrive at

$$
\frac{d\Sigma}{dp^2} = -\frac{\lambda^2}{3072\pi^4} \iiint dx\,dy\,dz\,\delta(x+y+z-1) \frac{\gamma}{(\alpha\beta)^2} \times \left\{ \frac{1}{\varepsilon} + 2\log \frac{\mu^2}{m^2} + C \right\} \tag{S.15}
$$

where $C = 2\log(4\pi) - 2\gamma_E$ is a numerical constant while $\alpha(x,y,z)$, $\beta(x,y,z)$ and $\gamma(x,y,z)$ depend on the Feynman parameters according to eqs. (S.2) and (S.4).

**Problem 4:**

We are left with one more task, namely integrating over the Feynman parameters. This looks like a daunting task, especially if one wants analytic dependence on the external momentum $p^2$, but fortunately we are only interested in one particular value of $p^2 = \text{physical mass}^2$. Since we are working at the leading order of perturbation theory which contributes to the $d\Sigma/dp^2$, we may neglect the difference between the physical and the bare masses as higher-order correction and set $p^2 = m^2$. Consequently, eq. (S.15) simplifies to

$$
\left. \frac{d\Sigma}{dp^2} \right|_{p^2=m^2} = -\frac{\lambda^2}{3072\pi^4} \iiint dx\,dy\,dz\,\delta(x+y+z-1) \frac{\gamma}{(\alpha\beta)^2} \times \left\{ \frac{1}{\varepsilon} + 2\log \frac{\mu^2}{m^2} + C \right\} + \log \frac{\alpha\beta}{(1-\gamma)^2} \tag{S.16}
$$

$$
= -\frac{\lambda^2}{3072\pi^4} \iiint dx\,dy\,dz\,\delta(x+y+z-1) \frac{xyz}{(xy+xz+yz)^3} \times \frac{x^2 y^2 z^2}{(xy+xz+yz)^3} \times \left\{ \frac{1}{\varepsilon} + 2\log \frac{\mu^2}{m^2} + C + \log \frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz)^2} \right\}
$$
where the second equality follows from eqs. (S.2) and (S.4) for the $\alpha(x, y, z)$, $\beta(x, y, z)$, and $\gamma(x, y, z)$.

Despite the above simplification, eq. (S.16) is a painful mess to evaluate. And that’s why I gave you eqs. (11) which tells you what the integrals actually are. Using eqs. (11), we find

$$\left. \frac{d\Sigma}{dp^2} \right|_{p^2=m^2} = -\frac{\lambda^2}{6144\pi^4} \left\{ \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C - \frac{3}{2} \right\}$$

(S.17)

to the leading order in $\lambda$, and therefore

$$Z = \frac{1}{1 - \left. \frac{d\Sigma}{dp^2} \right|_{p^2=M^2}} = 1 + \frac{\lambda^2}{6144\pi^4} \left\{ \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C - \frac{3}{2} \right\} + O(\lambda^3).$$

(S.18)