

Problem 1:

Our starting point is eq. (3) where $x + y + z \equiv 1$ and $q_3 = p - q_1 - q_2$. Expanding q_3^2 and collecting all the terms containing the q_1 momentum into a full square, we find

$$xq_1^2 + yq_2^2 + z(p - q_1 - q_2)^2 = (x + z) \left(q_1 + \frac{z}{x + z} (q_2 - p) \right)^2 + \frac{xz}{x + z} (q_2 - p)^2 + yq_2^2. \quad (\text{S.1})$$

Naturally, we interpret the first term here as αk_1^2 , thus

$$\alpha = (x + z), \quad k_1 = q_1 + \frac{z}{x + z} (q_2 - p). \quad (\text{S.2})$$

For the other two terms on the right hand side of eq. (S.1), we expand $(q_2 - p)^2$ and collect all terms containing the q_2 momentum into another full square, thus

$$\frac{xz}{x + z} (q_2 - p)^2 + yq_2^2 = \frac{xz + y(x + z)}{x + z} \left(q_2 - \frac{xz}{xz + y(x + z)} p \right)^2 + \frac{xzy}{xz + y(x + z)} p^2. \quad (\text{S.3})$$

Consequently, we define

$$\beta = \frac{xy + xz + yz}{x + z}, \quad \gamma = \frac{xyz}{xy + xz + yz}, \quad k_2 = q_2 - \frac{xz}{xy + xz + yz} p, \quad (\text{S.4})$$

which makes the right hand side of eq. (S.3) into $\beta k_2^2 + \gamma p^2$. Altogether, we have

$$xq_1^2 + yq_2^2 + zq_3^2 = \alpha k_1^2 + \beta k_2^2 + \gamma p^2 \quad (\text{S.5})$$

and hence eq. (4).

Finally, we need to check the Jacobian of replacing the original independent loop momenta q_1 and q_2 with k_1 and k_2 . In light of eqs. (S.2) and (S.4), it is easy to see that

$$\frac{\partial(k_1, k_2)}{\partial(q_1, q_2)} = \det \begin{pmatrix} 1 & \frac{z}{x+z} \\ 0 & 1 \end{pmatrix} = 1, \quad (\text{S.6})$$

and therefore $dk_1 dk_2 = dq_1 dq_2$, dimension by dimension. In other words, for fixed Feynman parameters,

$$\int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} = \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4}. \quad (\text{S.7})$$

Problem 2:

Assembling all the factors of the two loop amplitude (1) and making use of eqs. (2), (4), and (S.7), we have

$$\begin{aligned} -i\Sigma(p^2) &= \frac{(-i\lambda)^2}{3!} \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i0} \frac{i}{q_3^2 - m^2 + i0} \frac{i}{q_3^2 - m^2 + i0} \\ &= \frac{i\lambda^2}{6} \iiint dx dy dz \delta(x + y + z - 1) \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{2}{\mathcal{D}^3} \end{aligned} \quad (\text{S.8})$$

where \mathcal{D} is as in eq. (4). In particular, the dependence on the external momentum p comes solely through the γp^2 term in \mathcal{D} , hence

$$\frac{d\Sigma}{dp^2} = -\frac{\lambda^2}{6} \iiint dx dy dz \delta(x + y + z - 1) \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{(-6\gamma)}{\mathcal{D}^4}. \quad (\text{S.9})$$

Note that for large loop momenta k_1 and k_2 , \mathcal{D} grows like k^2 . Consequently, the integrand of the 8-dimensional momentum integral (S.8) behaves like $1/k^6$, so the integral diverges quadratically. On the other hand, the integrand of (S.9) behaves like $1/k^8$, so the divergence of this integral is only logarithmic.

Problem 3:

Rotating both loop momenta k_1 and k_2 into Euclidean momentum space, we have $d^4k_1 \rightarrow id^4k_1^E$, $d^4k_2 \rightarrow id^4k_2^E$, and

$$\mathcal{D} \rightarrow -\alpha(k_1^E)^2 - \beta(k_2^E)^2 + \gamma p^2 - m^2, \quad (\text{S.10})$$

hence

$$\frac{d\Sigma}{dp^2} = +\frac{\lambda^2}{6} \iiint dx dy dz \delta(x+y+z-1) \int \frac{d^4k_1^E}{(2\pi)^4} \int \frac{d^4k_2^E}{(2\pi)^4} \frac{6 \times (-\gamma)}{[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2]^4}. \quad (\text{S.11})$$

Next, we need dimensional regularization to actually perform the momentum integrals. Changing

$$\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \rightarrow \mu^{2(4-D)} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \quad (\text{S.12})$$

(Euclidean signature for all dimensions), we have

$$\begin{aligned} & \mu^{8-2D} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \frac{6}{[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2]^4} = \\ & \quad \langle\langle \text{using eq. (6)} \rangle\rangle \\ & = \mu^{8-2D} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \int_0^\infty dt t^3 \exp\left(-t[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2]\right) \\ & = \mu^{8-2D} \int_0^\infty dt t^3 e^{-t(m^2 - \gamma p^2)} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} e^{-t\alpha k_1^2} e^{-t\beta k_2^2} \\ & \quad \langle\langle \text{using eq. (7)} \rangle\rangle \\ & = \mu^{8-2D} \int_0^\infty dt t^3 e^{-t(m^2 - \gamma p^2)} \times (4\pi\alpha t)^{-D/2} (4\pi\beta t)^{-D/2} \\ & = \frac{\mu^{8-2D}}{(4\pi)^D (\alpha\beta)^{D/2}} \times \int_0^\infty dt t^{3-D} e^{-t(m^2 - \gamma p^2)} \\ & = \frac{\mu^{8-2D}}{(4\pi)^D (\alpha\beta)^{D/2}} \times \Gamma(4-D) (m^2 - \gamma p^2)^{D-4}. \end{aligned} \quad (\text{S.13})$$

Note the $\Gamma(4 - D)$ factor: It has a pole at $D = 4$ but no poles at $D < 4$. This is dimensional regularization's way to show that the momentum integrals diverge, but only logarithmically.

At this point, we may take $D = 4 - \epsilon$ for an infinitesimally small ϵ . Hence, the last line of eq. (S.13) becomes

$$\frac{1}{(4\pi)^4(\alpha\beta)^2} \Gamma(2\epsilon) \left(\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2 - \gamma p^2} \right)^{2\epsilon} \xrightarrow{\epsilon \rightarrow 0} \frac{1}{(4\pi)^4(\alpha\beta)^2} \times \left(\frac{1}{2\epsilon} - \gamma_E + \log \frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2 - \gamma p^2} \right). \quad (\text{S.14})$$

Plugging this formula back into eq. (S.11) and assembling all the factors, we finally arrive at

$$\frac{d\Sigma}{dp^2} = -\frac{\lambda^2}{3072\pi^4} \iiint dx dy dz \delta(x+y+z-1) \frac{\gamma}{(\alpha\beta)^2} \times \left\{ \begin{array}{l} \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C \\ + \log \frac{\alpha\beta}{[1 - (p^2/m^2)\gamma]^2} \end{array} \right\} \quad (\text{S.15})$$

where $C = 2 \log(4\pi) - 2\gamma_E$ is a numerical constant while $\alpha(x, y, z)$, $\beta(x, y, z)$ and $\gamma(x, y, z)$ depend on the Feynman parameters according to eqs. (S.2) and (S.4).

Problem 4:

We are left with one more task, namely integrating over the Feynman parameters. This looks like a daunting task, especially if one wants analytic dependence on the external momentum p^2 , but fortunately we are only interested in one particular value of $p^2 = \text{physical mass}^2$. Since we are working at the leading order of perturbation theory which contributes to the $d\Sigma/dp^2$, we may neglect the difference between the physical and the bare masses as higher-order correction and set $p^2 = m^2$. Consequently, eq. (S.15) simplifies to

$$\begin{aligned} \left. \frac{d\Sigma}{dp^2} \right|_{p^2=m^2} &= -\frac{\lambda^2}{3072\pi^4} \iiint dx dy dz \delta(x+y+z-1) \frac{\gamma}{(\alpha\beta)^2} \times \left\{ \begin{array}{l} \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C \\ + \log \frac{\alpha\beta}{(1-\gamma)^2} \end{array} \right\} \\ &= -\frac{\lambda^2}{3072\pi^4} \iiint dx dy dz \delta(x+y+z-1) \left(\frac{xyz}{(xy+xz+yz)^3} \right) \times \\ &\quad \times \left\{ \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C + \log \frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz)^2} \right\} \end{aligned} \quad (\text{S.16})$$

where the second equality follows from eqs. (S.2) and (S.4) for the $\alpha(x, y, z)$, $\beta(x, y, z)$, and $\gamma(x, y, z)$.

Despite the above simplification, eq. (S.16) is a painful mess to evaluate. And that's why I gave you eqs. (11) which tells you what the integrals actually are. Using eqs. (11), we find

$$\left. \frac{d\Sigma}{dp^2} \right|_{p^2=m^2} = -\frac{\lambda^2}{6144\pi^4} \left\{ \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C - \frac{3}{2} \right\} \quad (\text{S.17})$$

to the leading order in λ , and therefore

$$Z = \left. \frac{1}{1 - \frac{d\Sigma}{dp^2}} \right|_{p^2=M^2} = 1 + \frac{\lambda^2}{6144\pi^4} \left\{ \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C - \frac{3}{2} \right\} + O(\lambda^3). \quad (\text{S.18})$$