Problem **2**(a):

Let us start with the superficial degree of divergence $\mathcal{D}$. At large momenta, bosonic propagators behave as $1/q^2$ while fermionic propagators behave as $1/q$, hence in 4 dimensions

$$\mathcal{D} \;=\; 4L \;-\; 2P_B \;-\; P_F\,. \tag{S.1}$$

As in the $\lambda\phi^4$ theory, we can relate this expression to the numbers of external legs using the vertex valences. Naively, the Yukawa theory has only one vertex type — which connects one bosonic line and two fermionic lines — but we shall see that renormalization requires an additional four-boson vertex of the $\lambda\phi^4$ type. Denoting the respected numbers of the two vertex types V_Y and V_λ, we have

$$\begin{aligned} 2P_F \;+\; E_F \;&=\; 2V_Y\,,\\ 2P_B \;+\; E_B \;&=\; V_Y \;+\; 4V_\lambda\,, \end{aligned} \tag{S.2}$$

while the Euler formula says

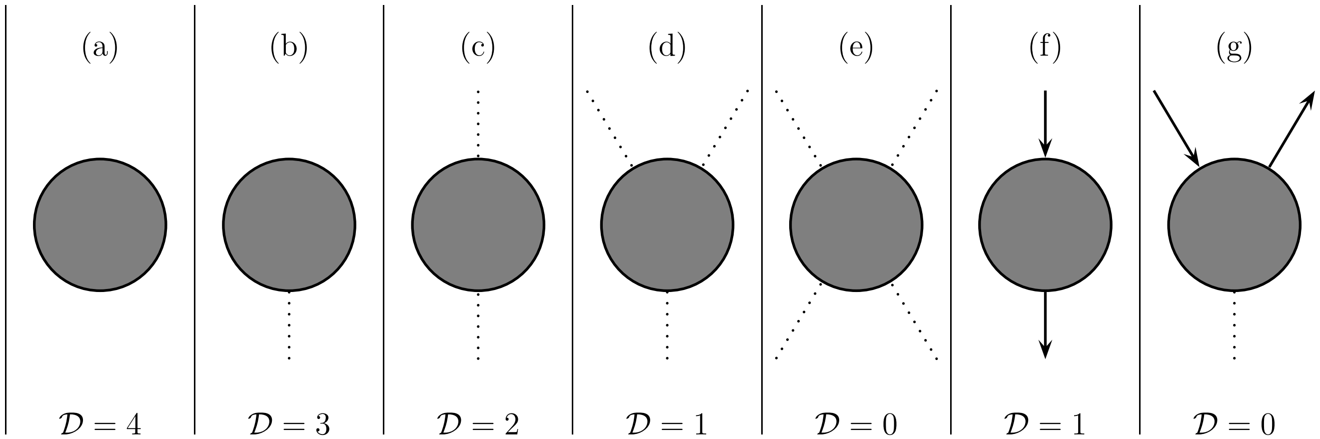
$$L \;-\; P \;+\; V \;\equiv\; L \;-\; P_B \;-\; P_F \;+\; V_Y \;+\; V_\lambda \;=\; 1, \tag{S.3}$$

Combining these three equations, we obtain

$$\begin{aligned} \mathcal{D} \;&=\; 4L - 2P_B - P_F \;=\; 4(L - P_B - P_F) \;+\; 3P_F \;+\; 2P_B\\ &=\; 4(1 - V_Y - V_\lambda) \;+\; \tfrac{3}{2}(2V_Y - E_F) \;+\; (V_Y + 4V_\lambda - E_B)\\ &=\; 4 \;-\; \tfrac{3}{2}E_F \;-\; E_B\,. \end{aligned} \tag{S.4}$$

Thus, the external legs of a diagram completely determine its superficial degree of divergence.

Consequently, for any number of loops, there are only seven superficially divergent amplitudes, namely



Furthermore, the amplitude (a) here is the vacuum energy while the amplitudes (b) and (d) vanish because of the parity symmetry. Indeed, the *pseudo*-scalar field Φ is parity-odd, hence the amplitudes involving odd number of pseudoscalar particles and no fermions must have parity-odd dependence on the particles' momenta. But to construct a parity-odd Lorentz-invariant combination of the Lorentz vectors $p_1^\alpha, p_2^\beta, \ldots$, one needs ϵ tensors, *e.g.* $\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta$, which requires at least 4 linearly independent momenta (in $d=4$ spacetime) and hence $n \geq 5$ external legs. For the amplitudes (b) and (d) involving one or three pseudoscalars only and no fermions, such construction is not available and the amplitudes vanish identically.

Unlike QED, the Yukawa theory does not give rise to Ward identities, so any 1PI amplitude that can diverge generally does. Hence, expanding the 1PI amplitudes (c), (e), (f), and (g) in powers of relevant momenta we find the following independent divergences:

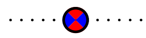
(c) $\Sigma_\phi(p^2) = O(\Lambda^2) \times \text{const} + O(\log\Lambda) \times p^2 + \text{finite};$

(e) $\mathcal{M}(s,t,u) = O(\log\Lambda) \times \text{const} + \text{finite};$

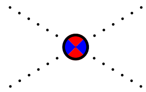
(f) $\Sigma_\psi(\not{p}) = O(\Lambda^1) \times \text{const} + O(\log\Lambda) \times \not{p} + \text{finite};$

(g) $\Gamma^5(p',p) = \gamma^5 \times O(\log\Lambda) \times \text{const} + \text{finite}.$

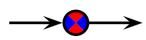
To cancel all these divergences *in situ* in the renormalized perturbation theory, we need four counterterm-related Feynman vertices, namely



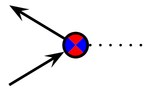
$$= \; -i\delta_m^{\phi} \; + \; ip^2\,\delta_Z^{\phi}\,,$$



$$= \; -i\delta_\lambda\,,$$



$$= \; -i\delta_m^{\psi} \; + \; i\,\not{p}\,\delta_Z^{\psi}\,,$$



$$= \; -\delta_g\gamma^5 \qquad \text{(S.5)}$$

Clearly, all these vertices follow from local (in x) counterterms in the renormalized Lagrangian, specifically

$$\mathcal{L}^{\text{counter}}_{\text{terms}} \; = \; \tfrac{1}{2}\delta_Z^{\phi}\,(\partial\Phi)^2 \; - \; \tfrac{1}{2}\delta_m^{\phi}\,\Phi^2 - \; \tfrac{1}{4!}\delta_\lambda\,\Phi^4 \; + \; i\delta_Z^{\psi}\,\overline{\Psi}\,\not{\partial}\Psi \; - \; \delta_m^{\psi}\,\overline{\Psi}\Psi \; - \; i\delta_g\,\Phi\overline{\Psi}\gamma^5\Psi. \qquad \text{(S.6)}$$

In order to produce such counterterms, one starts from the bare Lagrangian

$$\mathcal{L}_{\text{bare}} \; = \; \tfrac{1}{2}(\partial\Phi_0)^2 \; - \; \tfrac{1}{2}m_0^2\Phi_0^2 - \; \tfrac{1}{4!}\lambda_0\Phi_0^4 + \; \overline{\Psi}_0(i\,\not{\partial} - M_0)\Psi_0 \; - \; ig_0\,\Phi_0\overline{\Psi}_0\gamma^5\Psi_0\,, \qquad \text{(S.7)}$$

renormalizes the bare fields $\Phi_0(x) = \sqrt{Z_\phi}\Phi_r(x)$, $\Psi_0(x) = \sqrt{Z_\psi}\Psi_r(x)$, and splits the Lagrangian into

$$\mathcal{L}_{\text{bare}} \; = \; \mathcal{L}^{\text{phys}} \; + \; \mathcal{L}^{\text{counter}}_{\text{terms}} \qquad \text{(S.8)}$$

where

$$\mathcal{L}^{\text{phys}} \; = \; \tfrac{1}{2}(\partial\Phi_r)^2 \; - \; \tfrac{1}{2}m_{\text{phys}}^2\Phi_r^2 - \; \tfrac{1}{4!}\lambda_{\text{phys}}\Phi_r^4 + \; \overline{\Psi}_r(i\,\not{\partial} - M_{\text{phys}})\Psi_r \; - \; ig_{\text{phys}}\,\Phi_r\overline{\Psi}_r\gamma^5\Psi_r\,, \qquad \text{(S.9)}$$

the counterterms are exactly as in eq. (S.6) (where $\Phi \equiv \Phi_r$ and $\Psi \equiv \Psi_r$), and the coefficients are

$$\delta_Z^{\phi} = Z_\phi - 1, \qquad \delta_Z^{\psi} = Z_\psi - 1, \qquad \delta_m^{\phi} = Z_\phi m_0^2 - m_{\text{phys}}^2\,, \qquad \delta_m^{\psi} = Z_\psi M_0 - M_{\text{phys}},$$
$$\delta_\lambda = Z_\phi^2\lambda_0 - \lambda_{\text{phys}}\,, \qquad \text{and} \quad \delta_g = Z_\psi Z_\phi^{1/2} g_0 - g_{\text{phys}}\,.$$

We shall see momentarily that at the one-loop level of the theory we already need all the counterterms (S.6). In particular, we do need δ_λ even if we start with $\lambda_{\text{phys}} = 0$. Thus, from the bare Lagrangian point of view, $\lambda_{\text{phys}} = 0$ has no special meaning: $\lambda_0 \neq 0$ and vanishing of a particular scattering amplitude we use to define the physical λ would be just an accident. In other words, we may *fine tune* λ_0 to achieve $\lambda = 0$ just as we can fine tune λ_0 to achieve any other experimental value of the physical coupling, but it would not have any special meaning for the theory itself.

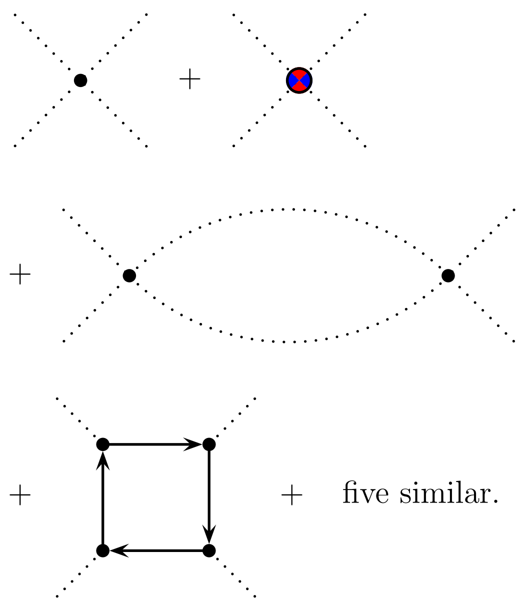
This is an example of the general rule: *barring fine tuning of the coupling parameters, a renormalizable quantum field theory has all the renormalizable couplings consistent with the theory's symmetries.* For the theory at hand, we have a Dirac field Ψ, a real pseudoscalar field Φ, and all the Lagrangian terms involving these fields should be invariant under Lorentz and parity transformations and have canonical dimensions ≤ 4 (for renormalizability's sake). There is only a finite number of such terms, and it is easy to see that the Lagrangian (S.9) comprises all such terms and no others. Consequently, the renormalized theory would not have any additional interactions.

Sometimes, in absence of some coupling the theory has an additional symmetry that would not be present otherwise. In such case, the extra symmetry would prevent such coupling from being restored by the renormalization procedure. For example, consider the Lagrangian (S.9) for $g = 0$ (but $\lambda \neq 0$): In the absence of the Yukawa coupling, the theory has an extra symmetry $\Phi(x) \to -\Phi(x)$ (without parity), and this extra symmetry would prevent the renormalization procedure from restoring the Yukawa coupling. On the other hand, when $\lambda = 0$ but $g \neq 0$, the theory does not has any additional symmetries it wouldn't have for $\lambda \neq 0$, and that's why the renormalization gives rise to the $\lambda\Phi^4$ coupling even if it wasn't there to begin with.

Problem **2**(b):

At this stage we are ready to calculate the counterterms, beginning with the δ_λ. At the one-loop level of analysis, the four-boson amplitude comprises the following Feynman diagrams:

$i\mathcal{M}^{1\,\text{loop}}(k_1, k_2, k_3, k_4) \;=$



$+$ two similar

$+$ five similar.

(S.10)

The last diagram here yields

$$-\int\frac{d^4p_1}{(2\pi)^4}\,\text{Tr}\left\{(-g\gamma^5)\,\frac{i}{\not{p}_1 - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{p}_2 - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{p}_3 - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{p}_4 - M + i0}\right\} \tag{S.11}$$

where

$$p_2 = p_1 + k_1\,, \qquad p_3 = p_2 + k_2, \qquad p_4 = p_3 + k_3, \quad \text{and} \quad p_1 = p_4 + k_4\,;$$

there are five similar diagrams related by permutations of the external momenta k_1, k_2, k_3, k_4. For generic values of these momenta, the integral (S.11) is quite complicated, but its divergence is k-independent and hence may be evaluated for any particular choice of k_i we find convenient. Clearly, the simplest set of k_i is $k_1 = k_2 = k_3 = k_4 = 0$; this is off-shell, but that's OK. Consequently, the integral (S.11) becomes

$$\begin{aligned} i\mathcal{V}^{\psi\,\text{loop}}(0,0,0,0) &= -\int\frac{d^4p_1}{(2\pi)^4}\,\text{tr}\left((-g\gamma^5)\,\frac{i}{\not{p}-M+i0}\right)^4 \\ &= -g^4\int\frac{d^4p_1}{(2\pi)^4}\,\frac{\text{tr}[\gamma^5(\not{p}+M)]^4}{(p^2-M^2+i0)^4} \\ &= \int\frac{d^4p_1}{(2\pi)^4}\,\frac{-4g^4}{(p^2-M^2+i0)^2} \end{aligned} \tag{S.12}$$

where the last equality follows from

$$[\gamma^5(\not{p}+M)]^2 \;=\; \gamma^5(\not{p}+M)\gamma^5(\not{p}+M) \;=\; (-\not{p}+M)(\not{p}+M) \;=\; -p^2\,+\,M^2 \tag{S.13}$$

and hence $\text{tr}[\gamma^5(\not{p}+M)]^4 = 4(p^2-M^2)^2$. Evaluating the integral on the last line of eq. (S.12) using dimensional regularization, we obtain

$$\mathcal{V}_{\psi\,\text{loop}}(k_1=k_2=k_3=k_4=0) \;=\; \frac{-4g^4}{16\pi^2}\left(\frac{1}{\bar{\epsilon}}+\log\frac{\mu^2}{M^2}\right) \tag{S.14}$$

where

$$\frac{1}{\bar{\epsilon}} \;\stackrel{\text{def}}{=}\; \frac{1}{\epsilon}\;-\gamma_E\;+\;\log(4\pi). \tag{S.15}$$

It remains to multiply the amplitude (S.14) by 6 (for six similar diagrams) and add contributions of the other diagrams (S.10). The latter diagrams have been evaluated in class in the context of the scalar $\lambda\Phi^4$ theory, thus to order $O(\lambda^2 \text{ or } g^4)$,

$$\mathcal{V}(k_1=k_2=k_3=k_4=0) \;=\; -\lambda\;-\;\delta_\lambda\;+\;\frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}}+\log\frac{\mu^2}{m^2}\right)\;-\;\frac{24g^4}{16\pi^2}\left(\frac{1}{\bar{\epsilon}}+\log\frac{\mu^2}{M^2}\right). \tag{S.16}$$

The renormalization condition for the physical λ coupling is the on-shell four-particle amplitude $\mathcal{M}(\text{threshold}) = -\lambda$, or in other words $\mathcal{V} = -\lambda$ when all external momenta are on shell and at the threshold ($s = 4m^2$, $t = u = 0$). At other values of external momenta, we should have

$$\mathcal{V}(k_1,k_2,k_3,k_4) \;=\; -\lambda \;-\; \frac{\lambda^2}{32\pi^2}\times\text{finite} \;-\; \frac{4g^4}{16\pi^2}\times\text{finite} \;+\; \text{higher loop orders.} \tag{S.17}$$

Comparing this formula with eq. (S.16) gives us

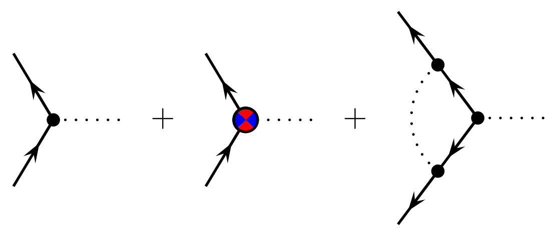
$$\delta_\lambda^{1\,\text{loop}} \;=\; \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\bar\epsilon} + \log\frac{\mu^2}{m^2} + \text{finite}\right) \;-\; \frac{24g^4}{16\pi^2}\left(\frac{1}{\bar\epsilon} + \log\frac{\mu^2}{M^2} + \text{finite}\right). \tag{S.18}$$

As promised, fermionic loops provide for $\delta_\lambda \neq 0$ even if were to start from $\lambda = 0$.

⋆ ⋆ ⋆

Next, we want to calculate the δ_g counterterm, so let us consider the $\Phi\overline{\Psi}\gamma^5\Psi$ vertex correction. By analogy with the QED vertex, we denote $\Gamma^{(5)}(p',p)$ the 1PI amplitude for two fermions of respective momenta p and p' and one pseudoscalar of momentum $k = p' - p$. At the one-loop level of analysis,

$$-\Gamma^{(5)}(p',p) \;=$$



(S.19)

$$\begin{aligned} &= -g\gamma^5 \;-\; \delta_g\gamma^5 \\ &\quad + \int\!\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2 - m^2 + i0}\times(-g\gamma^5)\,\frac{i}{\not{p}' + \not{q} - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{p} + \not{q} - M + i0}\,(-g\gamma^5). \end{aligned}$$

As in the previous calculation, the loop integral here diverges logarithmically, and the divergent part does not depend on the external momenta. Consequently, we may calculate this divergence for any values of p, p', and $k - p' - p$ we like, for example $p = p' = k = 0$ which makes for a much simpler integral. Indeed, for zero external momenta, the fermionic line becomes

$$\begin{aligned} (-g\gamma^5)\,\frac{i}{\not{q}' + \not{q} - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{q} + \not{q} - M + i0}\,(-g\gamma^5) \;&=\; g^3\,\frac{\gamma^5(\not{q} + M)\gamma^5(\not{Q} + M)\gamma^5}{(q^2 - M^2 + i0)^2} \\ &=\; g^3\,\frac{-\gamma^5}{(q^2 - M^2 + i0)^2} \end{aligned} \tag{S.20}$$

where the second equality follows from eq. (S.13). Consequently, the loop integral in eq. (S.19) becomes

$$-ig^3\gamma^5 \times \int\frac{d^4q}{(2\pi)^4}\,\frac{1}{(q^2-m^2+i0)(q^2-M^2+i0)} \;=\; +g^3\gamma^5\times\int\frac{d^4q_E}{(2\pi)^4}\,\frac{1}{(q^2+m^2)(q^2+M^2)}$$

⟪using dimensional regularization⟫

$$= +\frac{g^3\gamma^5}{16\pi^2}\left(\frac{1}{\bar\epsilon} \;+\; \int\limits_0^1 dx\,\log\frac{\mu^2}{xM^2+(1-x)m^2}\right) \tag{S.21}$$

and hence to the order g^3,

$$\Gamma^{(5)}(p'=p=0) \;=\; -g\gamma^5 \;-\; \delta_g\gamma^5 \;+\; \frac{g^3\gamma^5}{16\pi^2}\left(\frac{1}{\bar\epsilon} \;+\; \log\frac{\mu^2}{M^2} \;+\; \text{finite}\right). \tag{S.22}$$

And since the divergent part is momentum independent, it follows that for any external momenta,

$$\Gamma^{(5)}(p',p) \;=\; -g\gamma^5 \;-\; \delta_g\gamma^5 \;+\; \frac{g^3\gamma^5}{16\pi^2}\left(\frac{1}{\bar\epsilon} \;+\; \log\frac{\mu^2}{M^2} \;+\; \text{finite function of}(p',p)\right) \;+\; O(g^5 \text{ or } g^3\lambda). \tag{S.23}$$

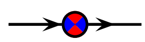
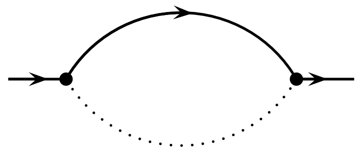
In class, I have not explained the renormalization condition for the Yukawa coupling g, but it's clear that such condition should have form $\Gamma^{(5)} = -g\gamma^5$ for the on-shell fermions and some particular value of the pseudoscalar's q^2, for example $q^2 = 0$ or on-shell $q^2 = m^2$ (allowed for $m \geq 2M$). In light of eq. (S.23), this means

$$\delta_g^{1\,\text{loop}} \;=\; \frac{g^3}{16\pi^2}\left(\frac{1}{\bar\epsilon} \;+\; \log\frac{\mu^2}{M^2} \;+\; \text{finite}\right) \tag{S.24}$$

where the finite part depends on the specific renormalization condition (and in general is a painfully complicated function of the m/M mass ratio) but the infinite part is clear and unambiguous.

⋆ ⋆ ⋆

Our next targets are the fermion's mass and kinetic energy counterterms δ_M^ψ and δ_Z^ψ. At the one-loop level of analysis, the Dirac field's 1PI two-point Green's function is

$-i\Sigma_\psi^{1\,\text{loop}}(\not{p}) \;=$  $+$ 

$$= \; -i\delta_M^\psi \;+\; i\delta_Z^\psi\,\not{p} \;+\int\!\frac{d^4k}{(2\pi)^4}\,\frac{i}{q^2-m^2+i0}\times(-g\gamma^5)\,\frac{i}{\not{p}+\not{q}-M+i0}\,(-g\gamma^5). \tag{S.25}$$

This time, we cannot set $p=0$ so we must be more careful. Let us re-write the integrand of the loop integral as

$$-g^2\,\frac{\mathcal{N}}{\mathcal{D}} \tag{S.26}$$

where

$$\mathcal{N} \;=\; \gamma^5(\not{q}+\not{p}+M)\gamma^5 \quad\text{and}\quad \mathcal{D} \;=\; \Bigl(q^2-m^2+i0\Bigr)\times\Bigl((p+q)^2-M^2+i0\Bigr). \tag{S.27}$$

Then using Feynman parameter trick we may further simplify the denominator

$$\begin{aligned}\frac{1}{\mathcal{D}} \;&=\; \int\limits_0^1\! dx\,\frac{1}{\bigl[(1-x)(q^2-m^2)\;+\;x((p+q)^2-M^2)\;+\;i0\bigr]^2}\\ &=\; \int\limits_0^1\! dx\,\frac{1}{\bigl[\ell^2\;-\;\Delta\;+\;i0\bigr]^2}\end{aligned} \tag{S.28}$$

where

$$\ell \;=\; q\;+\;xp \quad\text{and}\quad \Delta \;=\; xM^2\;+\;(1-x)m^2\;-\;x(1-x)p^2. \tag{S.29}$$

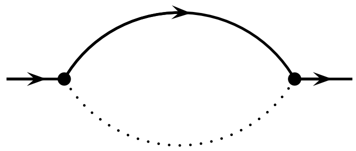
At the same time, in terms of the shifted loop momentum ℓ, the numerator becomes

$$\mathcal{N} \;=\; \gamma^5\Bigl(\not{\ell}\;+\;(1-x)\,\not{p}\;+\;M\Bigr)\gamma^5 \;=\; M\;-\;(1-x)\,\not{p}\;-\;\not{\ell}, \tag{S.30}$$

and furthermore the $\not{\ell}$ term does not contribute to the momentum integral because it's odd under the $\ell \to -\ell$ symmetry, thus

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\not{\ell}}{[\ell^2 - \Delta + i0]^2} \;=\; 0. \tag{S.31}$$

Altogether, this gives us the following expression for the loop integral in eq. (S.25):



$$= \; -g^2 \int_0^1 dx\, [M - (1-x)\,\not{p}] \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta + i0)^2}\,. \tag{S.32}$$

Curiously, this loop has superficial degree of divergence $D = +1$ but the actual momentum integral here diverges logarithmically rather than linearly. Evaluating this integral using dimensional regularization, we obtain

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta + i0)^2} \;=\; \frac{i}{16\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{\Delta}\right), \tag{S.33}$$

and therefore

$$\Sigma_\psi^{1\,\text{loop}}(\not{p}) \;=\; \delta_M^\psi \;-\; \delta_Z^\psi\,\not{p} + \frac{g^2}{16\pi^2}\int_0^1 dx\,[M - (1-x)\,\not{p}]\,\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{(1-x)m^2 + xM^2 - x(1-x)p^2}\right). \tag{S.34}$$

The renormalization conditions for the fermion's propagator correction $\Sigma^\psi(\not{p})$ are

$$\Sigma\bigg|_{\not{p}\,=\,M} \;=\; 0 \quad\text{and}\quad \frac{d\Sigma}{d\,\not{p}}\bigg|_{\not{p}\,=\,M} \;=\; 0. \tag{S.35}$$

In light of eq. (S.34), the second condition (S.35) becomes

$$\begin{aligned} \delta_Z^\psi[1\,\text{loop}] \;&=\; \frac{g^2}{16\pi^2}\frac{\partial}{\partial\,\not{p}}\int_0^1 dx\,[M - (1-x)\,\not{p}]\,\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{(1-x)m^2 + xM^2 - x(1-x)p^2}\right)\Bigg|_{\not{p}\,=\,M} \\ &=\; \frac{g^2}{16\pi^2}\int_0^1 dx\left[(x-1)\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{x^2M^2 + (1-x)m^2}\right) \;+\; \frac{2x^2(1-x)M^2}{x^2M^2 + (1-x)m^2}\right] \\ &=\; -\frac{g^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} \;+\; \log\frac{\mu^2}{M^2} \;+\; \text{finite}\right). \end{aligned} \tag{S.36}$$

At the same time, the first condition (S.35) implies

$$\begin{aligned}
\delta_M^\psi[1\,\text{loop}] \;&-\; M\delta_Z^\psi[1\,\text{loop}] \\
&= -\frac{g^2}{16\pi^2}\int\limits_0^1 dx\,[M-(1-x)\,\not{p}]\,\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{(1-x)m^2+xM^2-x(1-x)p^2}\right)\Bigg|_{\not{p}=M} \\
&= -\frac{g^2}{16\pi^2}\int\limits_0^1 dx\,xM\times\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{x^2M^2+(1-x)m^2}\right) \\
&= -\frac{g^2M}{32\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2}\,+\,\text{finite}\right)
\end{aligned} \tag{S.37}$$

and consequently

$$\delta_M^\psi[1\,\text{loop}] \;=\; -\frac{g^2M}{16\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2}\,+\,\text{finite}\right). \tag{S.38}$$

Note that similar to QED, the fermionic mass counterterm in the Yukawa theory is proportional to the mass itself and diverges logarithmically rather than linearly in the UV cutoff (*cf.* integral (S.32) prior to dimensional regularization). As in QED, this behavior is due an additional symmetry the Yukawa theory acquires when the fermion mass vanishes. Specifically, for $M=0$ we have a *discrete chiral symmetry*

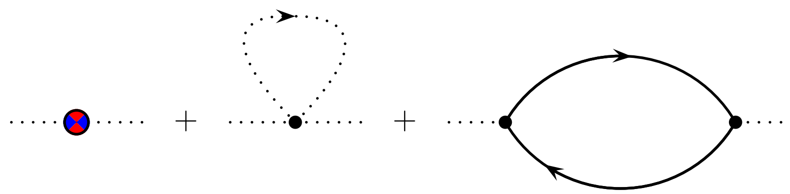
$$\Psi(x)\;\to\;\gamma^5\Psi(x),\quad \overline{\Psi}(x)\;\to\;-\overline{\Psi}(x)\gamma^5,\quad \Phi(x)\;\to\;-\Phi(x). \tag{S.39}$$

Unlike the gauge coupling in QED, the pseudoscalar Yukawa coupling does not respect continuous chiral transforms $\Psi(x)\to\exp(i\alpha\gamma^5)\Psi(x)$, but the discrete symmetry is sufficient for preventing the massless Yukawa theory from developing a mass shift via loop corrections.

⋆ ⋆ ⋆

Finally, consider the boson's mass and kinetic energy counterterms δ_M^ϕ and δ_Z^ϕ. At the one-loop level of analysis, the pseudoscalar field's 1PI two-point Green's function is

$$-i\Sigma_\phi^{1\,\rm loop}(k^2) \;=$$



$$= \; -i\delta_m^\phi \;+\; i\delta_Z^\phi\, k^2 \;+\; \frac{i\lambda m^2}{32\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,1\,+\,\log\frac{\mu^2}{m^2}\right) \\ -\int\!\frac{d^4p}{(2\pi)^4}\,{\rm tr}\left(\frac{i}{\not p - M + i0}\,(-g\gamma^5)\,\frac{i}{\not p + \not k - M + i0}\,(-g\gamma^5)\right). \tag{S.40}$$

Again, we re-write the fermionic loop integral as

$$+\,g^2\int\!\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}} \tag{S.41}$$

where the denominator is the usual

$$\mathcal{D} \;=\; \Big(p^2 - M^2 + i0\Big)\times\Big((p+k)^2 - M^2 + i0\Big) \tag{S.42}$$

and hence

$$\frac{1}{\mathcal{D}} \;=\; \int\limits_0^1\! dx\,\frac{1}{[\ell^2 - \Delta + i0]^2} \\ \text{for } \ell \;=\; p \;+\; kx \\ \text{and } \Delta \;=\; M^2 \;-\; x(1-x)k^2, \tag{S.43}$$

and the numerator is

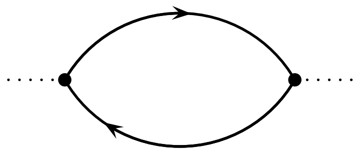
$$\begin{aligned} \mathcal{N} \;&=\; {\rm tr}\big[(\not p + M)\gamma^5(\not p + \not k + M)\gamma^5\big] \\ &=\; {\rm tr}\big[(M + \not p)(M - \not p - \not k)\big] \\ &=\; 4M^2 \;-\; 4p(p+k) \\ &=\; 4M^2 \;-\; 4(\ell - xk)(\ell + k - xk) \\ &=\; 4M^2 \;-\; 4\ell^2 \;+\; 4x(1-x)k^2 \;-4(1-2x)\ell\cdot k. \end{aligned} \tag{S.44}$$

Again, the last term here is odd with respect to $\ell \to -\ell$ and hence does not contribute to the $\int d^4\ell$

integral. Effectively,

$$\mathcal{N} \;\cong\; 4M^2 \;-\; 4\ell^2 \;+\; 4x(1-x)k^2$$

and hence the integral (S.41) becomes



$$\begin{aligned} &= 4g^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\, \frac{M^2 + x(1-x)k^2 - \ell^2}{(\ell^2 - \Delta + i0)^2} \\ &= 4ig^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4}\, \frac{2M^2 - \Delta + \ell_E^2}{(\ell_E^2 + \Delta)^2}\,. \end{aligned} \tag{S.45}$$

In four dimensions, the momentum integral (S.45) diverges quadratically. Hence, in dimensional regularization, we need to analytically continue from $D = 4$ Euclidean dimensions down to $D < 2$, evaluate the integral for $D < 2$, and only then continue back to $D = 4 - 2\epsilon$. Thus, working in the Euclidean momentum space, we have

$$\begin{aligned} \int \frac{d^4\ell}{(2\pi)^4}\, \frac{2M^2 - \Delta + \ell^2}{(\ell^2 + \Delta)^2} \;&\longrightarrow\; \mu^{4-D} \int \frac{d^D\ell}{(2\pi)^D}\, \frac{2M^2 - \Delta + \ell^2}{(\ell^2 + \Delta)^2} \\ &= \mu^{4-D} \int \frac{d^D\ell}{(2\pi)^D} \int_0^\infty dt\, t\, e^{-t\Delta} \left(2M^2 - \Delta - \frac{\partial}{\partial t}\right) e^{-t\ell^2} \\ &= \mu^{4-D} \int_0^\infty dt\, t\, e^{-t\Delta} \left(2M^2 - \Delta - \frac{\partial}{\partial t}\right) \left[\int \frac{d^D\ell_E}{(2\pi)^D}\, e^{-t\ell_E^2} \;=\; (4\pi t)^{-D/2}\right] \\ &= \frac{\mu^{4-D}}{(4\pi)^{D/2}} \int_0^\infty dt\, t\, e^{-t\Delta} \left((2M^2 - \Delta)t^{-(D/2)} \;+\; \tfrac{D}{2}\, t^{-(D/2)-1}\right) \\ &= \frac{\mu^{4-D}}{(4\pi)^{D/2}} \left((2M^2 - \Delta)\Gamma(2 - \tfrac{D}{2})\Delta^{(D/2)-2} \;+\; \tfrac{D}{2}\Gamma(1 - \tfrac{D}{2})\Delta^{(D/2)-1}\right) \\ &\langle\!\langle \text{now take } D = 4 - 2\epsilon \rangle\!\rangle \\ &= \frac{1}{16\pi^2}\, \Gamma(\epsilon) \left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon} \left(2M^2 \;-\; \Delta \;+\; \frac{2-\epsilon}{\epsilon - 1}\Delta\right) \\ &\underset{\epsilon\to 0}{\longrightarrow}\; \frac{i}{16\pi^2} \left[(2M^2 - 3\Delta)\left(\frac{1}{\bar{\epsilon}} \;+\; \log\frac{\mu^2}{\Delta}\right) \;-\; \Delta\right]. \end{aligned} \tag{S.46}$$

Consequently,

$$\begin{aligned}\Sigma_\phi^{1\,\text{loop}}(k^2) \;=\; \delta_m^\phi \;-\; \delta_Z^\phi\, k^2 \;-\; \frac{\lambda m^2}{32\pi^2}\left(\frac{1}{\bar\epsilon} \,+\, 1 \,+\, \log\frac{\mu^2}{m^2}\right)\\ -\; \frac{g^2}{4\pi^2}\int\limits_0^1 dx\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right) \;-\; \Delta\right].\end{aligned} \tag{S.47}$$

Similarly to the fermion's propagator correction Σ_ψ discussed above, the renormalization conditions for a scalar or a pseudoscalar field are

$$\Sigma_\phi\Big|_{k^2=m^2} \;=\; 0 \qquad \text{and} \qquad \frac{\partial\Sigma_\phi}{\partial k^2}\Big|_{k^2=m^2} \;=\; 0. \tag{S.48}$$

Therefore, in light of eq. (S.47),

$$\begin{aligned}\delta_Z^\phi[1\,\text{loop}] \;&=\; -\frac{g^2}{4\pi^2}\frac{\partial}{\partial k^2}\int\limits_0^1 dx\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\;-\;\Delta\right]_{k^2=m^2}\\ &=\; +\frac{g^2}{4\pi^2}\int\limits_0^1 dx\,x(1-x)\times\frac{\partial}{\partial\Delta}\left((2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\;-\;\Delta\right)\Big|_{k^2=m^2}\\ &=\; -\frac{g^2}{4\pi^2}\frac{\partial}{\partial k^2}\int\limits_0^1 dx\,x(1-x)\times\left[\frac{3}{\bar\epsilon}\;+\;3\log\frac{\mu^2}{M^2-x(1-x)k^2}\;+\;\frac{2x(1-x)k^2}{M^2-x(1-x)k^2}\right]_{k^2=m^2}\\ &=\; -\frac{g^2}{8\pi^2}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{M^2}\;+\;\text{finite}\right).\end{aligned} \tag{S.49}$$

Likewise,

$$\begin{aligned}&\delta_m^\phi[1\,\text{loop}] \;-\; m^2\,\delta_Z^\phi[1\,\text{loop}] \;-\; \frac{\lambda m^2}{32\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,1\,+\,\log\frac{\mu^2}{m^2}\right)\\ &\quad=\; -\frac{g^2}{4\pi^2}\int\limits_0^1 dx\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\;-\;\Delta\right]_{k^2=m^2}\\ &\quad=\; -\frac{g^2}{4\pi^2}\int\limits_0^1 dx\left[\Big(3x(1-x)m^2-M^2\Big)\times\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2-x(1-x)m^2}\right)+\;\text{finite}\right]\\ &\quad=\; -\frac{g^2}{4\pi^2}\left((\tfrac{1}{2}m^2-M^2)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2}\right)+\;\text{finite}\right]\end{aligned} \tag{S.50}$$

and hence

$$\delta_m^\phi[1\,\text{loop}] \;=\; \left[\frac{\lambda m^2}{32\pi^2} \;+\; \frac{g^2 M^2}{4\pi^2}\right] \times \left(\frac{1}{\bar\epsilon} \;+\; \log\frac{\mu^2}{M^2}\right) \;+\; \text{finite}. \tag{S.51}$$

Note that unlike the other counterterms of the Yukawa theory, the pseudoscalar mass correction δ_m^ϕ diverges quadratically rather than logarithmically. The dimensional regularization however does not see the quadratic divergence itself, all it sees is the sub-leading logarithmic divergence accompanying the quadratic divergence. Thus, in terms of a different UV cutoff, eq. (S.51) means

$$\delta_m^\phi[1\,\text{loop}] \;=\; (\text{unknown}) \times \Lambda^2 \;+\; \left[\frac{\lambda m^2}{32\pi^2} \;+\; \frac{g^2 M^2}{4\pi^2}\right] \times \log\frac{\Lambda^2}{M^2} \;+\; \text{finite}, \tag{S.52}$$

where the coefficient of the leading Λ^2 divergence depends on the cutoff's details — such as the exact definition of Λ^2 for each cutoff. FYI, for the Wilson's hard-edge cutoff

$$\delta_m^\phi[1\,\text{loop}] \;=\; -\frac{\lambda}{32\pi^2}\left(\Lambda^2 \;-\; m^2\log\frac{\Lambda^2}{m^2}\right) \;-\; \frac{g^2}{4\pi^2}\left(\Lambda^2 \;-\; M^2\log\frac{\Lambda^2}{M^2}\right) \;+\text{finite}. \tag{S.53}$$