Problem **1** (problem 1(d) from the previous set):

At the end of solution for part (b) we saw that un-renormalized gauge invariance of the bare Lagrangian requires $Z_1' = Z_1 = Z_2$. Equivalently, in terms of the renormalized perturbation theory, we can say that *the counterterms should be invariant under the un-modified gauge symmetry of the physical Lagrangian*, and this requires

$$\begin{aligned}\delta_2 \times \partial^\mu\Phi^*\,\partial_\mu\Phi \;+\; ie\delta_1 \times A^\mu(\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*) \;+\; e^2\delta_1' \times A^\mu A_\mu \Phi^*\Phi \;\propto \\ \propto\; D^\mu\Phi^*\,D_\mu\Phi \;\equiv\; (\partial^\mu\Phi^* + ieA^\mu\Phi^*)(\partial_\mu\Phi - ieA^\mu\Phi)\end{aligned} \tag{S.1}$$

and hence

$$\delta_2 \;=\; \delta_1 \;=\; \delta_1'\,. \tag{S.2}$$

In class, we proved a similar equality $\delta_1 = \delta_2$ for the ordinary QED using the Ward–Takahashi identity

$$k_\mu\Gamma^\mu(p',p) \;=\; \big(\not p' - m - \Sigma(\not p')\big) \;-\; \big(\not p - m - \Sigma(\not p)\big) \tag{S.3}$$

for the 1PI amplitudes we have derived from the WT identity

$$k_\mu S^\mu(k;p',p) \;=\; eS(p') \;-\; eS(p) \tag{S.4}$$

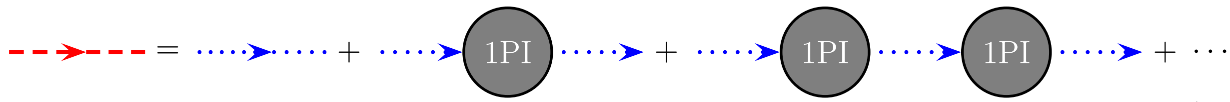
for the un-amputated amplitudes. In an earlier homework (set #15) we proved similar WT identities for the un-amputated amplitudes of the scalar QED. Our present task is to use these identities to derive analogues of eq. (S.3) and then prove eqs. (S.2).

Our starting point is a pair of Ward–Takahashi identities for the un-amputated two-scalar amplitudes, namely

$$k_\mu\,S^\mu(k;p',p) \;=\; eS(p') \;-\; eS(p), \tag{S.5}$$

$$k_{1\mu}\,S^{\mu\nu}(k_1,k_2;p'p) \;=\; eS^\nu(k_2;p',p+k_1) \;-\; eS^\nu(k_2;p'-k_1,p). \tag{S.6}$$

Diagrammatically, the photon-less amplitude $S(p)$ is the dressed scalar propagator,



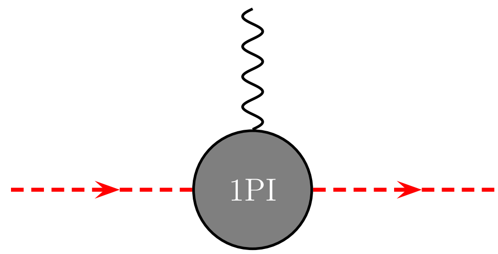
(S.7)

and hence

$$S(p) \;=\; \frac{i}{p^2 - M^2 - \Sigma(p^2)}\,. \tag{S.8}$$

The one-photon amplitude $S^\mu(k;p',p)$ is amputated with respect to the photon but not the charged scalars. Diagrammatically, summing over the leg bubbles on external scalar lines gives us two dressed propagators, thus

$S^\mu(k;p',p) \;=$



$= \; S(p') \times ieG^\mu(k;p',p) \times S(p).$ (S.9)

Applying the WT identity (S.5) to this amplitude gives us

$$S(p') \times iek_\mu \times G^\mu(k.p',p) \times S(p) \;=\; eS(p') \;-\; eS(p) \tag{S.10}$$

and hence

$$\begin{aligned} k_\mu \times G^\mu(k.p',p) \;&=\; \frac{i}{S(p')} \;-\; \frac{i}{S(p)} \\ &=\; \big(p'^2 - M^2 - \Sigma(p'^2)\big) \;-\; \big(p^2 - M^2 - \Sigma(p^2)\big). \end{aligned} \tag{S.11}$$

Or rather

$$k_\mu \times G^\mu_{\text{tree+loops}}(k;p',p) \;=\; \big(p'^2 - M^2 - \Sigma_{\text{loops}}(p'^2)\big) \;-\; \big(p^2 - M^2 - \Sigma_{\text{loops}}(p^2)\big) \tag{S.12}$$

because the identity (S.5) was derived for the bare theory without accounting for the counterterms.

In part (a) of this problem (*cf.* previous homework) we used eq. (S.12) to argue that the divergence of the $G^\mu_{\rm loops}(k;p',p)$ must be proportional to the $(p'+p)^\mu$. This time, we shall use it to relate the renormalization conditions for the δ_1 and δ_2 counterterms. Indeed, consider the right hand of eq. (S.12) in the limit of small photon's momentum $k\to 0$ and hence $p^{'} \approx p^2$. In this limit

$$\begin{aligned}\big(p'^2-M^2-\Sigma_{\rm loops}(p'^2)\big)\ -\ \big(p^2-M^2-\Sigma_{\rm loops}(p^2)\big)\ &=\\ &=\ (p'^2-p^2)\times\left(1\,-\,\frac{d\Sigma_{\rm loops}}{dp^2}\right)\ +\ O\big((p'^2-p^2)^2\big)\\ &=\ k_\mu\times(p'+p)^\mu\left(1\,-\,\frac{d\Sigma_{\rm loops}}{dp^2}\right)\ +\ O(|k|^2),\end{aligned} \tag{S.13}$$

therefore

$$G^\mu_{\rm tree+loops}(k=0,p'=p)\,+\ =\ (p'+p)^\mu\times\left(1\,-\,\frac{d\Sigma_{\rm loops}}{dp^2}\right), \tag{S.14}$$

and hence, in light of $G^\mu_{\rm tree}=(p'+p)^\mu$,

$$G^\mu_{\rm loops}(p'=p)\ =\ (p'+p)^\mu\times -\frac{d\Sigma_{\rm loops}}{dp^2}\,. \tag{S.15}$$

Now consider the renormalization conditions for the δ_1 and δ_2 counterterms. As explained in part (c) of the problem,

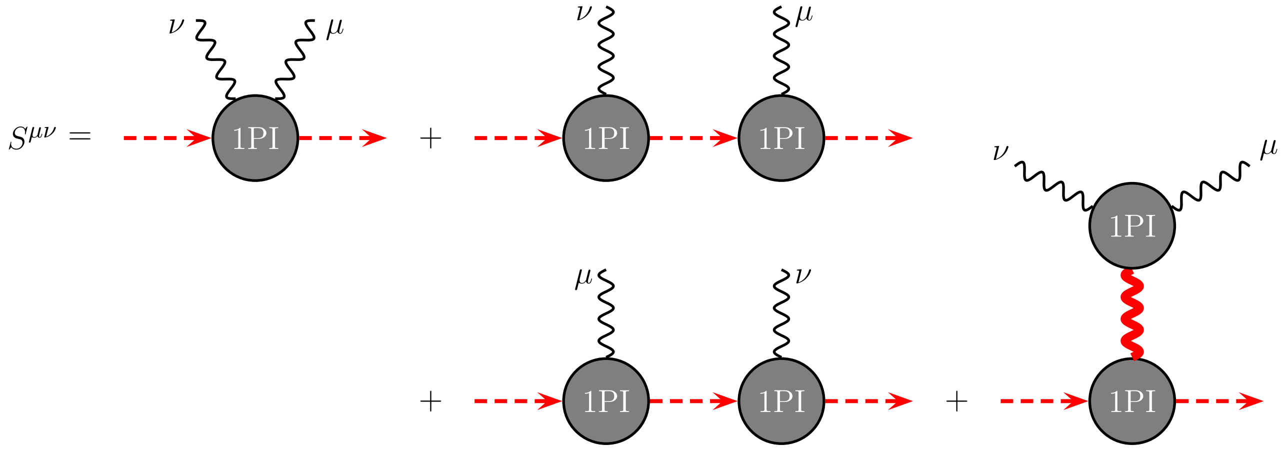
$$\delta_2\ =\ \left.\frac{d\Sigma_{\rm loops}}{dp^2}\right|_{p^2=M^2}, \tag{S.16}$$

$$\delta_1\times(p+p')^\mu\ =\ -G^\mu_{\rm loops}(k=0,p'=p,p^2=M^2). \tag{S.17}$$

Therefore, putting $p=p'$ on shell in eq. (S.15), we immediately obtain

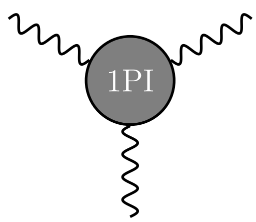
$$\delta_1\ =\ \delta_2\,. \tag{S.18}$$

To prove the second relation (S.2) we need the two-photon amplitude $G^{\mu\nu}(k_1,k_2;p',p)$. Again, this is an un-amputated amplitude comprised of an amputated core and two dressed scalar propagators. However, in this case the amputated core is not necessarily one-particle irreducible: Instead, it may comprise two 1PI sub-diagrams connected by a dressed propagator. Altogether, we have four distinct diagram topologies



(S.19)

but fortunately the last topology does not contribute because its top 1PI sub-amplitude vanishes by charge-conjugation symmetry,



$$= \; 0 \tag{S.20}$$

Spelling out the other three topologies in terms of dressed *scalar* propagators and 1PI one-photon and two-photon amplitudes, we have

$$\begin{aligned} S^{\mu\nu}(k_1,k_2;p',p) \;&=\; S(p') \times ie^2 G^{\mu\nu}(k_1,k_2;p',p) \times S(p) \qquad\qquad \text{(S.21)} \\ &+\; S(p') \times ieG^{\nu}(k_2;p',p+k_1) \times S(p+k_1) \times ieG^{\mu}(k_1;p+k_1,p) \times S(p) \\ &+\; S(p') \times ieG^{\mu}(k_1;p',p'-k_1) \times S(p'-k_1) \times ieG^{\nu}(k_2;p'-k_1,p) \times S(p). \end{aligned}$$

Let us multiply both sides of eq. (S.21) by $k_{1\mu}$. On the left had side, we use WT identity (S.6) and obtain

$$\begin{aligned} k_{1\mu} \times S^{\mu\nu}(k_1, k_2; p', p) \;&=\; eS^\nu(k_2; p', p+k_1) \;-\; eS^\nu(k_2; p'-k_1, p) \\ &=\; eS(p') \times ieG^\nu(k_2; p', p-k_1) \times S(p+k_1) \\ &\quad -\; eS(p'-k_1) \times ieG^\nu(k_2; p'-k_1, p) \times S(P). \end{aligned} \tag{S.22}$$

On the right hand side of eq. (S.21) we use WT identity (S.5) (in the form of eq. (S.10)) for the last two terms, hence

$$\begin{aligned} k_{1\mu} \times S^{\mu\nu}(k_1, k_2; p', p) \;=\;& S(p') \times ie^2 k_{1\mu} \times G^{\mu\nu}(k_1, k_2; p', p) \times S(p) \\ &+\; S(p') \times ieG^\nu(k_2; p', p+k_1) \times \big[eS(p+k_1) \;-\; eS(p)\big] \\ &+\; \big[eS(p') \;-\; eS(p'-k_1)\big] \times ieG^\nu(k_2; p'-k_1, p) \times S(p). \end{aligned} \tag{S.23}$$

Equating the last two formulæ and working through the algebra, we arrive at a fairly simple Ward–Takahashi identity for the 1PI amplitudes, namely

$$k_{1\mu} \times G^{\mu\nu}(k_1, k_2; p', p) \;=\; G^\nu(k_2; p', p+k_1) \;-\; G^\nu(k_2; p'-k_1, p). \tag{S.24}$$

Or rather

$$k_{1\mu} \times G^{\mu\nu}_{\text{tree+loops}}(k_1, k_2; p', p) \;=\; G^\nu_{\text{tree+loops}}(k_2; p', p+k_1) \;-\; G^\nu_{\text{tree+loops}}(k_2; p'-k_1, p) \tag{S.25}$$

because we started with un-amputated WT identities for the bare amplitudes without the counterterms.

It is easy to see that the tree-level $G^{\mu\nu}_{\text{tree}} = 2g^{\mu\nu}$ and $G^\nu_{\text{tree}}(k; p', p) = (p'+p)^\nu$ satisfy eq. (S.25) all by themselves. Consequently, the 1PI loop amplitudes satisfy their own identity

$$k_{1\mu} \times G^{\mu\nu}_{\text{loops}}(k_1, k_2; p', p) \;=\; G^\nu_{\text{loops}}(k_2; p', p+k_1) \;-\; G^\nu_{\text{loops}}(k_2; p'-k_1, p). \tag{S.26}$$

It is this identity which assures counterterm equality $\delta_1' = \delta_1 = \delta_2$. To see how this works, we need to take both photons' momenta to zero, but it's convenient to set $k_2 \to 0$ first and only then take $k_1 \to 0$. For $k_2 = 0$ and hence $p' = p + k_1$, eq. (S.26) becomes

$$\begin{aligned} k_{1\mu} \times G^{\mu\nu}_{\text{loops}}(k_1, 0; p', p) \; &= \; G^{\nu}_{\text{loops}}(0; p', p') \; - \; G^{\nu}_{\text{loops}}(0; p, p) \\ &= \; -2p'^{\nu} \, \frac{d\Sigma_{\text{loops}}(p'^2)}{dp'^2} \; + \; 2p^{\nu} \, \frac{d\Sigma_{\text{loops}}(p^2)}{dp^2} \end{aligned} \tag{S.27}$$

where the second equality here follows from eq. (S.15). Now we can take $k_1 \to 0$ and hence $p' \to p$, and in this limit the bottom line of eq. (S.27) becomes

$$k_{1\mu} \times \frac{\partial}{\partial p_\mu} \left[-2p^{\nu} \, \frac{d\Sigma_{\text{loops}}(p^2)}{dp^2} \right] \; + \; O(|k_1|^2) \tag{S.28}$$

Consequently,

$$G^{\mu\nu}_{\text{loops}}(0, 0; p, p) \; = \; \frac{\partial}{\partial p_\mu} \left[-2p^{\nu} \, \frac{d\Sigma_{\text{loops}}(p^2)}{dp^2} \right] \; = \; g^{\mu\nu} \times \mathcal{A}_{\text{loops}}(p^2) \; + \; p^{\mu} p^{\nu} \times \mathcal{P}_{\text{loops}}(p^2) \tag{S.29}$$

where

$$A_{\text{loops}}(p^2) \; = \; -2 \frac{d\Sigma_{\text{loops}}(p^2)}{dp^2} \,, \qquad B_{\text{loops}}(p^2) \; = \; +4 \frac{d^2\Sigma_{\text{loops}}(p^2)}{(dp^2)^2} \,. \tag{S.30}$$

As explained in part (c) of the problem, the renormalization condition for the δ_1' counterterm reads

$$\delta_1' \; = \; -\tfrac{1}{2} \mathcal{A}_{\text{loops}}(p^2 = M^2) \tag{S.31}$$

where $\mathcal{A}$ is exactly as in eq. (S.29). Therefore, in light of eq. (S.30),

$$\delta_1' \; = \; + \left. \frac{d\Sigma_{\text{loops}}}{p^2} \right|_{p^2 = M^2} \; = \; \delta_2 \,. \tag{S.32}$$

And this (almost) completes our proof of eqs. (S.2).

However, we still have a loophole to close. The loop amplitudes $\Sigma_{\text{loops}}(p^2)$, $G^{\mu}_{\text{loops}}(k; p', p)$, and $G^{\mu\nu}_{\text{loops}}(k_1, k_2; p', p)$ involved in renormalization conditions for the δ_2, δ_1, and δ_1' counterterms include both pure loops of the bare theory, but also loops containing some counterterm vertices of lower order in α. On the other hand, the Ward–Takahashi identities (S.5) and (S.6) were derived in the homework #15 for the bare theory, without the counterterms. It's fairly easy to extend that derivation to the renormalized theory with the counterterms, but that requires gauge invariance of the counterterm Lagrangian. In other words, we must assume eqs. (S.2) in order to prove the WT identities we have used here to prove eqs. (S.2).

In order to break this logical circle, we expand the counterterms in powers of the couplings $\alpha = e^2/4\pi$ and λ, and prove eqs. (S.2) by induction in the loop order. As the base of the induction, we notice that at the one-loop level of the perturbative expansion, the loop amplitudes are pure loops and do not contain the counterterm vertices. Consequently, at this level, the WT identities (S.5) and (S.6) must hold true, and hence our proof of eqs. (S.2) is valid at the one-loop level.

Now, suppose we have proven eqs. (S.2) to the n–loop level. At the $n+1$ loop order, the loop diagrams may contain counterterm vertices of orders n or less; by induction assumption, these vertices are gauge invariant, and hence the $n+1$ loop amplitudes do satisfy the WT identities. Consequently, our proof of eqs. (S.2) is valid to the $n+1$ loop order, and this gives us the induction step.

Therefore, by induction, eqs. (S.2) hold true to all orders of the perturbation theory. $\mathcal{Q.E.D.}$.

Problem **2**:

Please download the latest version of class notes. The printed notes distributed in class had several bad typos in equations.

Problem **3**(a):

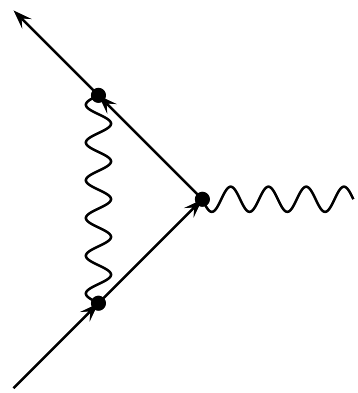
As explained in class, the net electric form factor of the electron

$$F_1^{\rm net}(q^2) \;=\; 1^{\rm tree} \;+\; F_1^{\rm loops}(q^2) \;+\; \delta_1 \tag{S.33}$$

must be exactly $=1$ for $q^2=0$ and hence

$$\delta_1 \;=\; -F_1^{\rm loops}(q^2=0). \tag{S.34}$$

At the one-loop level, the QED vertex correction comes from a single diagram



(S.35)

and hence, as explained in detail in the supplementary note (*cf.* problem (2)),

$$F_1^{1\,\text{loop}}(q^2) \;=\; -2ie^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{\mu^{4-D}\,d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_1}{\left[\ell^2-\Delta+i0\right]^3} \tag{S.36}$$

where

$$\begin{aligned}\mathcal{N}_1 \;\cong\; \frac{(D-2)^2}{D}\,\ell^2 \;-\; 2(1-4z+z^2)\,m^2 \;-\; 2(z+xy)q^2 \\ +\; (4-D)((1-z)^2m^2+xyq^2)\end{aligned} \tag{S.37}$$

and

$$\Delta \;=\; (xp+yp')^2 \;+\; x(m^2-p^2) \;+\; y(m^2-p'^2) \;=\; (1-z)^2m^2 \;-\; xyq^2 \quad \langle\!\langle\text{on shell}\rangle\!\rangle. \tag{S.38}$$

Note that for $q^2 = 0$, both $\mathcal{N}_1$ and Δ depend on only one Feynman parameter, namely z. Therefore, integrating over x and y, we obtain

$$\delta_1^{1\,\text{loop}} \;=\; +2ie^2 \int_0^1 dz\,(1-z) \int \frac{\mu^{4-D}d^D\ell}{(2\pi)^D}\,\frac{\frac{(D-2)^2}{D}\,\ell^2 \;-\; \left[2(1-4z+z^2)-(4-D)(1-z)^2\right]m^2}{\left[\ell^2-\Delta+i0\right]^3} \tag{S.39}$$

Now, let's integrate over the loop momentum ℓ. In D Minkowski dimensions, we have

$$\begin{aligned}
\int\!\frac{\mu^{4-D}\,d^D\ell}{(2\pi)^D}\,\frac{1}{\left[\ell^2-\Delta+i0\right]^3} &= -i\mu^{4-D}\int\!\frac{d^D\ell_E}{(2\pi)^D}\,\frac{1}{\left[\ell_E^2+\Delta\right]^3} \\
&= -i\mu^{4-D}\int\!\frac{d^D\ell_E}{(2\pi)^D}\left[\frac{1}{2}\int_0^\infty\!dt\,t^2\,e^{-t(\ell_E^2+\Delta)}\right] \\
&= -\frac{i}{2}\mu^{4-D}\int_0^\infty\!dt\,t^2\,e^{-t\Delta}\int\!\frac{d^D\ell_E}{(2\pi)^D}\,e^{-t\ell_E^2} \\
&= -\frac{i}{2}\,\frac{\mu^{4-D}}{(4\pi)^{D/2}}\int_0^\infty\!dt\,t^{2-\frac{D}{2}}\,e^{-t\Delta} \\
&= -\frac{i}{2}\,\frac{\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(3-\frac{D}{2}\right)\,\Delta^{\frac{D}{2}-3} \\
&= \frac{-i}{32\pi^2}\,\Gamma(1+\epsilon)\,\frac{(4\pi\mu^2)^\epsilon}{\Delta^{1+\epsilon}}
\end{aligned} \tag{S.40}$$

where on the last line we let $D=4-2\epsilon$ but don't assume that ϵ is small. Likewise,

$$\begin{aligned}
\int\!\frac{\mu^{4-D}\,d^D\ell}{(2\pi)^D}\,\frac{\ell^2}{\left[\ell^2-\Delta+i0\right]^3} &= +i\mu^{4-D}\int\!\frac{d^D\ell_E}{(2\pi)^D}\,\frac{\ell_E^2}{\left[\ell_E^2+\Delta\right]^3} \\
&= i\mu^{4-D}\int\!\frac{d^D\ell_E}{(2\pi)^D}\,\ell_E^2\times\left[\frac{1}{2}\int_0^\infty\!dt\,t^2\,e^{-t(\ell_E^2+\Delta)}\right] \\
&= \frac{i}{2}\mu^{4-D}\int_0^\infty\!dt\,t^2\,e^{-t\Delta}\times\left(-\frac{\partial}{\partial t}\right)\int\!\frac{d^D\ell_E}{(2\pi)^D}\,e^{-t\ell_E^2} \\
&= \frac{i}{2}\,\frac{\mu^{4-D}}{(4\pi)^{D/2}}\int_0^\infty\!dt\,t^2\,e^{-t\Delta}\times\left(-\frac{\partial}{\partial t}\,t^{-\frac{D}{2}}\;=\;+\frac{D}{2}\,t^{-1-\frac{D}{2}}\right) \\
&= \frac{i}{2}\,\frac{\mu^{4-D}}{(4\pi)^{D/2}}\times\frac{D}{2}\,\Gamma\left(2-\frac{D}{2}\right)\,\Delta^{\frac{D}{2}-2} \\
&= \frac{i}{32\pi^2}\,(2-\epsilon)\Gamma(\epsilon)\,\frac{(4\pi\mu^2)^\epsilon}{\Delta^\epsilon}\,.
\end{aligned} \tag{S.41}$$

The momentum integral in eq. (S.39) is clearly a linear combination of the two above integrals with z–dependent coefficients, therefore

$$\delta_1 \;=\; -\frac{e^2}{16\pi^2}\,(4\pi\mu^2)^\epsilon \int\limits_0^1 dz\,(1-z) \begin{bmatrix} \dfrac{\Gamma(\epsilon)}{\Delta^\epsilon} \times 2(1-\epsilon)^2 + \\ +\; \dfrac{\Gamma(1+\epsilon)}{\Delta^{1+\epsilon}} \times \bigl(2(1-4z+z^2) - 2\epsilon(1-z)^2\bigr) \end{bmatrix}. \qquad \text{(S.42)}$$

Next, we need to integrate over the Feynman parameter z. According to eq. (S.38), $\Delta = (1-z)^2 m^2$ for $q^2 = 0$. Substituting this formula into eq. (S.42) gives us

$$\begin{aligned} \delta_1 \;&=\; -\frac{\alpha}{4\pi}\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \Gamma(\epsilon) \times \int\limits_0^1 dz\,(1-z)^{1-2\epsilon}\left[2(1-\epsilon)^2 \;+\; 2\epsilon \times \frac{(1-4z+z^2)-\epsilon(1-z)^2}{(1-z)^2}\right] \\ &=\; -\frac{\alpha}{4\pi}\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \Gamma(\epsilon) \times \left[(1-z)^{2-2\epsilon} \;+\; \frac{4\epsilon}{1-2\epsilon}\,(1-z)^{1-2\epsilon} \;-\; 2(1-z)^{-2\epsilon}\right]_{z=1}^{z=0} \end{aligned} \qquad \text{(S.43)}$$

and for $\epsilon < 0$ — *i.e.*, for $D > 4$ — this integral has a finite value

$$\delta_1 \;=\; -\frac{\alpha}{4\pi}\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \Gamma(\epsilon) \times \left[3 \;+\; \frac{4\epsilon}{1-2\epsilon}\right]. \qquad \text{(S.44)}$$

However. for $D < 4$ and hence $\epsilon > 4$, the last term on the second line of eq. (S.43) diverges for $z \longrightarrow 1$.

By rules of the dimensional regularization, we can evaluate the integrals in any dimension which make them convergent and then analytically continue to any other D. Unfortunately, in the present situation convergence of the momentum integral requires $D < 4$ (*cf.* eq. (S.41)) while convergence of the Feynman-parameter needs $D > 4$. The two demands are clearly inconsistent, and hence for any dimension D the Feynman diagram (S.35) is divergent one way or another. Consequently, the dimensional regularization is not enough, and we need an additional regulator to make the diagram converge for *some* dimension.

To understand the nature of the divergences, let us go back to the Feynman diagram (S.35) and write the amplitude as an integral over the un-shifted loop momentum k:

$$ie\gamma^{\mu}_{1\,\mathrm{loop}} = e^3 \int \frac{\mu^{4-d}\,dk^D}{(2\pi)^D}\,\frac{\mathcal{N}^{\mu}}{\mathcal{D}} \tag{S.45}$$

where the numerator

$$\begin{aligned}\mathcal{N}^{\mu} &= \gamma^{\nu}(\not{k} + \not{p}' + m)\gamma^{\mu}(\not{k} + \not{p} + m)\gamma_{\nu} \\ &= \begin{cases} O(k^2) & \text{for } k \to \infty, \\ \text{finite} & \text{for } k \to 0, \end{cases}\end{aligned} \tag{S.46}$$

while the denominator

$$\begin{aligned}\mathcal{D} &= [k^2 + i0] \times [(p' + k)^2 - m^2 + i0] \times [(k + p)^2 - m^2 + i0] \\ &= [k^2 + i0] \times [k^2 + 2kp' + i0] \times [k^2 + 2kp + i0] \\ &= \begin{cases} k^6 & \text{for } k \to \infty, \\ O(k^4) & \text{for } k \to 0 \end{cases}\end{aligned} \tag{S.47}$$

(assuming the external electron legs are on-shell, $p^2 = p^{2\prime 2} = m^2$). Consequently,

$$\frac{\mathcal{N}_{\mu}}{\mathcal{D}} \propto \frac{1}{k^4} \tag{S.48}$$

both for $k \to 0$ and for $k \to \infty$, and hence the $\int d^D k$ integral diverges in all dimensions. Specifically, it has both the ultraviolet and the infra-red divergences in four dimensions, and changing the dimension cures one divergence but makes the other worse.

To solve this problem, we need two separate regulators, one UV and one IR. The dimensional regularization can serve as either UV or IR regulator (by taking either $D < 4$ or $D > 4$) but not both.

The problem tells us to use DR as a UV regulator and a tiny photon mass as a IR regulator. We assume that somehow such mass is consistent with a Feynman gauge for the photon's propagator and so make the latter simply

$$= \frac{-ig^{\mu\nu}}{k^2 - m_\gamma^2 + i0} \tag{1}$$

without the additional $i\frac{k^\mu k^\nu}{m_\gamma^2}$ term in the numerator. Hence, in eq. (S.45) the numerator $\mathcal{N}^\mu$ remains exactly as in eq. (S.46) while the denominator becomes

$$\mathcal{D} \;=\; [k^2 - m_\gamma^2 + i0] \times [k^2 + 2kp' + i0] \times [k^2 + 2kp + i0]. \tag{S.49}$$

And therefore, the $F_1^{1\,\mathrm{loop}}(q^2)$ form factor remains exactly as in eq. (S.36) where the numerator is exactly as in eq. (S.37) but in the denominator we now have

$$\Delta \;=\; (xp+yp')^2 \,+\, zm_\gamma^2 \;=\; (1-z)^2 m_e^2 \,-\, xyq^2 \,+\, zm_\gamma^2 \;\xrightarrow[q^2=0]{}\; (1-z)^2 m_e^2 \,+\, zm_\gamma^2\,. \tag{S.50}$$

For the present purposes, this means that the δ_1 counterterm is given by eq. (S.42) where Δ is as in eq. (S.50). Using $\Gamma(1+\epsilon) = \epsilon\Gamma(\epsilon)$ and a bit of algebra to simplify the integrand of eq. (S.42), we arrive at

$$\delta_1^{1\,\mathrm{loop}} \;=\; -\frac{\alpha}{4\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon} \times \int_0^1\! dz\, \frac{2(1-\epsilon)(1-z)^3 \,+\, 4\epsilon(1-z)^2 \,-\, 4\epsilon(1-z) \,+\, O(m_\gamma^2/m_e^2)}{\big[(1-z)^2 \,+\, (m_\gamma^2/m_e^2)z\big]^{1+\epsilon}}\,. \tag{S.51}$$

The photon mass (acting as an IR regulator) is supposed to be tiny, which for our purposes means $m_\gamma \ll m_e$. Hence, we may neglect the $O(m_\gamma^2/m_e^2)$ terms in the numerator, while in the denominator the $(m_\gamma^2/m_e^2)z$ term becomes important only when $z \approx 1$. Thus, we approximate

$$\begin{aligned} \delta_1^{1\,\mathrm{loop}} \;&\approx\; -\frac{\alpha}{4\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon} \times \int_0^1\! dz \left[\begin{aligned} &2(1-\epsilon)(1-z)^{1-2\epsilon} \,+\, 4\epsilon(1-z)^{-2\epsilon} \,- \\ &\qquad -\, 4\epsilon(1-z) \times \big[(1-z)^2 + (m_\gamma^2/m_e^2)\big]^{-1-\epsilon} \end{aligned}\right] \\ &=\; -\frac{\alpha}{4\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon} \times \left[\begin{aligned} &(1-z)^{2-2\epsilon} \,+\, \frac{4\epsilon}{1-2\epsilon}\,(1-z)^{1-2\epsilon} \\ &\qquad +\, 2\big[(1-z)^2 + (m_\gamma^2/m_e^2)\big]^{-\epsilon} \end{aligned}\right]_{z=1}^{z=0}. \end{aligned} \tag{S.52}$$

This integral converges for $D > 3$ and yields

$$\delta_1^{\text{1 loop}} \;=\; -\frac{\alpha}{4\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon} \times \left[1 \;+\; \frac{4\epsilon}{1-2\epsilon} \;+\; 2 \;-\; 2\left(\frac{m_e^2}{m_\gamma^2}\right)^{\epsilon}\right]. \qquad \text{(S.53)}$$

Unlike eq. (S.44) which was based on two contradictory assumption, eq. (S.53) is valid formula for a finite range of dimensions $3 < D < 4$, and this allows analytic continuation to any complex D. For most purposes, we need only the $D \to 4$ limit

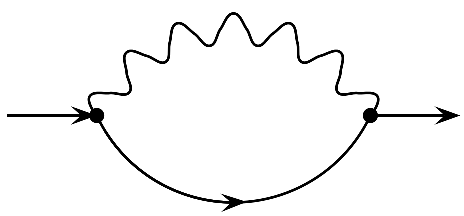
$$\delta_1^{\text{1 loop}} \;\xrightarrow[D\to 4]{}\; -\frac{\alpha}{4\pi}\left(\frac{1}{\epsilon} \;-\; \gamma_E \;+\; \log\frac{4\pi\mu^2}{m_e^2} \;+\; 4 \;-\; 2\log\frac{m_e^2}{m_\gamma^2}\right), \qquad \text{(S.54)}$$

but to make sure $\delta_1 = \delta_2$ in any dimension (*cf.* part (c) of this problem), we need eq. (S.53) as it is.

<u>Problem **3**(b)</u>:

Now consider the electron's propagator correction $\Sigma(\not{p})$. At the one-loop level of analysis,

$$-i\Sigma^{\text{1 loop}}(\not{p}) \;=$$



$$= \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\lambda\nu}}{k^2 - m_\gamma^2 + i0} \times ie\gamma_\lambda\,\frac{i}{\not{k} + \not{p} - m_e + i0} \times ie\gamma_\nu \qquad \text{(S.55)}$$

$$= \; -e^2 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}}$$

where the numerator is

$$\mathcal{N} \;=\; \gamma^\nu(\not{k} + \not{p} + m_e)\gamma_\nu \;=\; Dm_e \;-\; (D-2)(\not{p} + \not{k}) \qquad \text{(S.56)}$$

and the denominator is

$$\frac{1}{\mathcal{D}} \;=\; \frac{1}{k^2-m_\gamma^2+i0}\times\frac{1}{(k+p)^2-m_e^2+i0} \;=\int\limits_0^1\!dz\,\frac{1}{(\ell^2-\Delta+i0)^2} \tag{S.57}$$

for

$$\ell^2\;-\;\Delta\;=\;z(k^2-m_\gamma^2)\;+\;(1-z)((k+p)^2-m_e^2) \tag{S.58}$$

and hence

$$\ell\;=\;k\;+\;(1-z)p, \tag{S.59}$$

$$\Delta\;=\;(1-z)m_e^2\;-\;z(1-z)p^2\;+\;zm_\gamma^2\,. \tag{S.60}$$

As usual, we re-express the numerator (S.56) in terms of the shifted loop momentum ℓ and then discard odd powers of ℓ, thus

$$\mathcal{N}\;=\;Dm_e\;-\;(D-2)(\not\ell+z\not p)\;\cong\;Dm_e\;-\;(D-2)z\not p, \tag{S.61}$$

and therefore

$$\Sigma^{1\,\text{loop}}(\not p)\;=\;-ie^2\int\limits_0^1\!dz\,\Big(Dm_e\;-\;(D-2)z\not p\Big)\times\int\limits_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i0)^2}\,. \tag{S.62}$$

The momentum integral here should be familiar to you by now, so let me simply state the result: In dimensional regularization

$$\begin{aligned}\int\limits_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i0)^2}\;&=\int\frac{\mu^{4-D}\,d^D\ell}{(2\pi)^D}\,\frac{1}{(\ell^2-\Delta+i0)^2}\\ &=\;\frac{i\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(2-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-2}\\ &=\;\frac{i}{16\pi^2}\,\Gamma(\epsilon)\,\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}.\end{aligned} \tag{S.63}$$

Consequently, eq. (S.62) evaluates to

$$\Sigma^{1\,\mathrm{loop}}(\not{p}) \;=\; \frac{\alpha}{4\pi}\,\Gamma(\epsilon)(4\pi\mu^2)^\epsilon \int_0^1 dz\,\frac{(4-2\epsilon)m_e \;-\; (2-2\epsilon)z\,\not{p}}{\Delta^\epsilon(z)}\,. \tag{S.64}$$

In the absence of the IR regulator (*i.e.*, for $m_gamma^2 = 0$), the integral (S.64) converges for $\epsilon < 1$ (*i.e.*, for $D > 2$) and off-shell momenta $p^2 < m_e^2$. Consequently, $\Sigma(\not{p})$ can be analytically continued to any complex D and $\not{p}$ and the IR regulator seems un-necessary. Unfortunately, this continuation has a mild (for small ϵ) singularity for $p^2 = m_e^2$, and consequently the derivative $d\Sigma/d\not{p}$ becomes infinite on shell. Indeed, taking the derivative of eq. (S.64) with respect to $\not{p}$, we have

$$\frac{d\Sigma^{1\,\mathrm{loop}}}{d\not{p}} \;=\; \frac{\alpha}{2\pi}\,\Gamma(\epsilon)(4\pi\mu^2)^\epsilon \int_0^1 dz\,\left(\frac{-(1-\epsilon)z}{\Delta^\epsilon} \;-\; \epsilon\frac{(2-\epsilon)m_e \;-\; (1-\epsilon)z\,\not{p}}{\Delta^{1+\epsilon}} \times \frac{\partial\Delta}{\partial\not{p}}\right) \tag{S.65}$$

where

$$\frac{\partial\Delta}{\partial\not{p}} \;=\; -2z(1-z)\,\not{p} \tag{S.66}$$

Let us neglect the IR regulator for a moment and take so $m_\gamma^2 = 0 \;\Rightarrow\; \Delta = (1-z)(m^2 - zp^2)$. Then for $z \approx 1$, the integrand of eq. (S.65) behaves as

$$(1-z)^{-\epsilon} \times \left[\frac{1-\epsilon}{[m^2 - zp^2]^\epsilon} \;+\; \frac{2\epsilon\,\not{p}((2-\epsilon)m - (1-\epsilon)\,\not{p})}{[m^2 - zp^2]^{1+\epsilon}}\right]. \tag{S.67}$$

For off-shell momenta $p^2 < m^2$, the expression in the square brackets here is finite and the $\int dz(1-z)^{-\epsilon}$ is perfectly finite as long as $\epsilon < 1$ *i.e.*, $D > 2$. But for the on-shell $p^2 = m^2$, the second term in the square brackets blows up at $z = 1$, we end up with

$$\int_0^1 dz\,\frac{\text{finite}}{(1-z)^{1+2\epsilon}}\,, \tag{S.68}$$

which diverges for any $\epsilon \geq 0$ *i.e.*, $D \leq 4$. And that's why we need the IR regulator $m_\gamma^2 > 0$.

So let us put the IR regulator back in business and calculate the counterterm

$$\delta_1^{1\,\mathrm{loop}} \;=\; \left.\frac{d\Sigma^{1\,\mathrm{loop}}}{d\,\not{p}}\right|_{\not{p}=m_e}. \tag{S.69}$$

For the on-shell external momentum $\not{p} = m_e$, the integral (S.65) simplifies to

$$\delta_2^{1\,\mathrm{loop}} \;=\; \frac{\alpha}{2\pi}\,\Gamma(\epsilon)(4\pi\mu^2)^\epsilon \int_0^1 dz\,\left(\frac{-(1-\epsilon)z}{\Delta^\epsilon} \;+\; \frac{2\epsilon z(1-z)((2-\epsilon)-(1-\epsilon)z)m^2}{\Delta^{1+\epsilon}}\right) \tag{S.70}$$

where

$$\Delta \;=\; (1-z)^2 m_e^2 \;+\; z m_\gamma^2 \;\approx\; (1-z)^2 m_e^2 \;+\; m_\gamma^2\,. \tag{S.71}$$

The approximation here follows from the IR regulator being important only for $z \approx 1$, and it allows us to re-write the integrand of (S.72) (S.70) as

$$m_e^{-2\epsilon}(1-z)^{-2\epsilon} \times \left(-(1-\epsilon)z \;+\; \frac{2\epsilon(1-z)}{(1-z)^2 + (m_\gamma^2/m_e^2)} \times \left(1-\epsilon(1-z)-(1-\epsilon)(1-z)^2\right)\right), \tag{S.73}$$

which after some algebra becomes

$$m_e^{-2\epsilon}(1-z)^{-2\epsilon} \times \left((1-\epsilon)(1-2\epsilon)(1-z) \;-\; (1-\epsilon+2\epsilon^2) \;+\; \frac{2\epsilon(1-z)}{(1-z)^2+(m_\gamma^2/m_e^2)}\right).$$

Consequently, we evaluate the integral (S.70) as

$$\begin{aligned}
\delta_2^{1\,\mathrm{loop}} \;&=\; \frac{\alpha}{2\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \int_0^1 dz \left[\begin{array}{r} (1-\epsilon)(1-2\epsilon)(1-z)^{1-2\epsilon} \;-\; (1-\epsilon+2\epsilon^2)(1-z)^{-2\epsilon} \\ +\; \dfrac{2\epsilon(1-z)}{\left[(1-z)^2+(m_\gamma^2/m_e^2)\right]^{1+\epsilon}} \end{array}\right] \\
&=\; \frac{\alpha}{2\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \times \left[\begin{array}{r} \dfrac{1-2\epsilon}{2}(1-z)^{2-2\epsilon} \;-\; \dfrac{1-\epsilon+2\epsilon^2}{1-2\epsilon}(1-z)^{1-2\epsilon} \\ -\; \left[(1-z)^2+(m_\gamma^2/m_e^2)\right]^{-\epsilon} \end{array}\right]_{z=1}^{z=0}
\end{aligned} \tag{S.74}$$

Thanks to the IT regulator, this integral converges for *epsilon* $< \frac{1}{2}$ and yields

$$\begin{aligned} \delta_2^{1\,\text{loop}} &= \frac{\alpha}{2\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon} \times \left[\frac{1-2\epsilon}{2} \;-\; \frac{1-\epsilon+2\epsilon^2}{1-2\epsilon} \;-\; 1 \;+\; \left(\frac{m_e^2}{m_\gamma^2}\right)^{\epsilon}\right] \\ &= \frac{\alpha}{2\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon} \times \left[-\frac{1}{2} \;-\; \frac{2\epsilon}{1-2\epsilon} \;-\; 1 \;+\; \left(\frac{m_e^2}{m_\gamma^2}\right)^{\epsilon}\right]. \end{aligned} \tag{S.75}$$

Note that for a finite range of dimensions $3 < D < 4$ all integrals leading to eq. (S.75) converge and both the UV and the IR divergences are regularized at the same time. This allows us to analytically continue eq. (S.75) to any complex D. However, for most physical purposes, we need only the $D \to 4$ limit of the counterterm

$$\delta_2^{1\,\text{loop}} \;\xrightarrow[D\to 4]{}\; -\frac{\alpha}{4\pi}\left(\frac{1}{\epsilon} \;-\; \gamma_E \;+\; \log\frac{4\pi\mu^2}{m_e^2} \;+\; 4 \;-\; 2\log\frac{m_e^2}{m_\gamma^2}\right). \tag{S.76}$$

Problem **3**(c):

A quick glance at eq. (S.53) and (S.75) reveals that indeed $\delta_1^{1\,\text{loop}} = \delta_2^{1\,\text{loop}}$ for any dimension D.

To be precise, we have proven that $\delta_1^{1\,\text{loop}} = \delta_2^{1\,\text{loop}}$ in the limit of $(m_\gamma^2/m_e^2) \to 0$. In fact, this limit is not necessary, and to see that we need to compare eqs. (S.42) and (S.70). Both equations use the same $\Delta(z) = (1-z)^2 m_e^2 + z m_\gamma^2$ and consequently have similar denominators. The numerators do not look similar, but with a bit of algebra and integration by parts, one can show that the two integrals are actually equal to each other. This work is left as a residual exercise for the students.