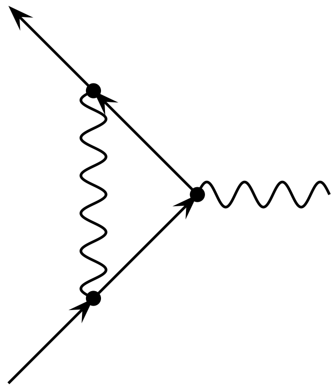
Problem **1**(a):

The one-loop QED vertex correction comes from a single Feynman diagram



(S.1)

Evaluating this diagram using gauge (1) for the internal photon propagator, we have

$$\begin{aligned}
ie\Gamma^\mu_{1\,\text{loop}}(p',p) \;&=\; \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\, ie\gamma_\nu \times \frac{i}{\not p' + \not k - m + i0} \times ie\gamma^\mu \times \frac{i}{\not p + \not k - m + i0} \times ie\gamma_\lambda \times \\
&\qquad\qquad \times \frac{-i}{k^2+i0}\left(g^{\nu\lambda} \;+\; (\xi-1)\frac{k^\nu k^\lambda}{k^2+i0}\right) \\
&=\; e^3 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2+i0} \times \gamma_\nu\, \frac{1}{\not p' + \not k - m + i0}\, \gamma^\mu \frac{1}{\not p + \not k - m + i0} \gamma^\nu \\
&\quad +\; (\xi-1)\times e^3 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\, \frac{1}{(k^2+i0)^2} \times \not k\, \frac{1}{\not p' + \not k - m + i0}\, \gamma^\mu\, \frac{1}{\not p + \not{\;}\! - m + i0}\; \not k .
\end{aligned} \tag{S.2}$$

Note that the third line here gives the whole one-loop vertex correction in the Feynman gauge, so the fourth line amounts to the gauge-dependent difference. In other words,

$$\begin{aligned}
ie\Gamma^\mu_\xi(p',p) \;-\; ie\Gamma^\mu_F(p',p) \;&=\; \\
=\; (\xi-1)\times e^3 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\, &\frac{1}{(k^2+i0)^2} \times \not k\, \frac{1}{\not p' + \not k - m + i0}\, \gamma^\mu\, \frac{1}{\not p + \not k - m + i0}\; \not k .
\end{aligned} \tag{S.3}$$

Now, let us simplify the integrand here. Using eq. (3) we re-write the two right-most factors as

$$\frac{1}{\not p + \not k - m + i0} \times \not k \;=\; 1 \;-\; \frac{1}{\not p + \not k - m + i0} \times (\not p - m) \;\cong\; 1 \tag{S.4}$$

because the second term on the right hand side vanishes in the on-shell context of $\bar{u}(p')\,\Gamma^\mu\,u(p)$ — note that $(\not p - m) \times u(p) = 0$. Likewise, using $\bar{u}(p') \times (\not p' - m) = 0$ we have for two other factors of the integrand

$$\not k \times \frac{1}{\not p' + \not k - m + i0} \;=\; 1 \;-\; (\not p' - m) \times \frac{1}{\not p' + \not k - m + i0} \;\cong 1, \tag{S.5}$$

and hence

$$\not k \, \frac{1}{\not p' + \not k - m + i0} \, \gamma^\mu \, \frac{1}{\not p + \not k - m + i0} \, \not k \;\cong\; \gamma^\mu \tag{S.6}$$

without any fermionic propagators at all. Consequently, eq. (S.3) simplifies to

$$ie\Gamma^\mu_\xi(p',p) \;-\; ie\Gamma^\mu_F(p',p) \;=\; (\xi - 1) \times e^3 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4} \, \frac{\gamma^\mu}{(k^2 + i0)^2} \tag{S.7}$$

or equivalently

$$\Gamma^\mu_\xi(p',p) \;=\; \Gamma^\mu_F(p',p) \;+\; e^2(\xi - 1)\gamma^\mu \times \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4} \, \frac{-i}{(k^2 + i0)^2} \tag{2}$$

Quod Ergo Demonstratum.

Problem **1**(b):

In terms of the electromagnetic form factors, eq. (2) means

$$\begin{aligned} F_2^{1\,\text{loop}}(q^2)[\xi] \;&=\; F_2^{1\,\text{loop}}(q^2)[\text{Feynman}], \\ \text{but } F_1^{1\,\text{loop}}(q^2)[\xi] \;&=\; F_1^{1\,\text{loop}}(q^2)[\text{Feynman}] \;+\; e^2(\xi - 1) \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4} \, \frac{-i}{(k^2 + i0)^2} \,. \end{aligned} \tag{S.8}$$

However, these equations refer to the pure loop-diagram contributions to the form factors and don't include the counterterms. The *net* form-factors are given by

$$\begin{aligned} F_2^{\text{net}}(q^2) \;&=\; F_2^{\text{loops}}(q^2), \\ F_1^{\text{net}}(q^2) \;&=\; 1^{\text{tree}} \;+\; F_1^{\text{loops}}(q^2) \;+\; \delta_1\,, \end{aligned} \tag{S.9}$$

where the δ_1 counterterm is set to

$$\delta_1 \;=\; -\; F_1^{\text{loops}}\Big|_{q^2=0} \tag{S.10}$$

to make sure that the net charge of the electron is not affected by the vertex corrections. Operationally, the δ_1 counterterm subtracts away the vertex corrections for $q^2 = 0$ so that only the momentum-dependent part of the $F_1^{\text{loops}}(q^2)$ affects the net form factor.

Now, let us look at the gauge-dependent contribution

$$e^2(\xi - 1)\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2} \tag{S.11}$$

to the $F_1^{1\,\text{loop}}(q^2)$. The integrand here does not depend on any of the external momenta. Moreover, the integral is over the un-shifted photon's momentum k, so the regulating the ultra-violet and the infra-red divergences of the integral would not introduce q^2–dependence through the back door. Thus, without specifying the UV and the IR regulators and actually evaluating the integral, we can say that its value — whatever it is — does not depend on q^2. Therefore, this momentum-independent but gauge-dependent contribution to the *net* $F_1(q^2)$ form factor is canceled out by the gauge-dependent adjustment to the δ_1 counterterm. Thus, altogether

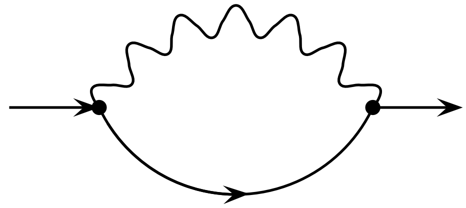
$$F_2^{\text{net}}(q^2)[\xi] \;=\; F_2^{\text{net}}(q^2)[\text{Feynman}], \tag{S.12}$$

$$F_1^{\text{net}}(q^2)[\xi] \;=\; F_1^{\text{net}}(q^2)[\text{Feynman}], \tag{S.13}$$

$$\text{but}\quad \delta_1^{1\,\text{loop}}[\xi] \;=\; \delta_1^{1\,\text{loop}}[\text{Feynman}] \;-\; e^2(\xi-1)\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2}\,. \tag{4}$$

<u>Problem **1**(c)</u>:

Now consider the electron's propagator correction $\Sigma(\not p)$. At the one-loop level there is only one diagram



(S.14)

hence using gauge (1) for the photon's propagators we obtain

$$\begin{aligned}
-i\Sigma^{1\,\text{loop}}_{\xi}(\not{p}) \;&=\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,ie\gamma_\lambda\,\frac{i}{\not{k}+\not{p}-m_e+i0}\times ie\gamma_\nu\times\frac{-i}{k^2+i0}\left(g^{\nu\lambda}\;+\;(\xi-1)\frac{k^\nu k^\lambda}{k^2+i0}\right)\\
&=\;-e^2\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+i0}\times\gamma^\nu\,\frac{1}{\not{k}+\not{p}-m_e+i0}\,\gamma_\nu\\
&\quad-\;(\xi-1)\times e^2\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+i0)^2}\times\not{k}\,\frac{1}{\not{k}+\not{p}-m_e+i0}\;\not{k}.
\end{aligned}\tag{S.15}$$

Again, the first term on the right hand side (the second line) here gives the entire propagator correction in the Feynman gauge, so the second term (the third line) is the gauge-dependent difference, thus,

$$\Sigma^{1\,\text{loop}}_{\xi}(\not{p})\;-\;\Sigma^{1\,\text{loop}}_{F}(\not{p})\;=\;(\xi-1)\times -ie^2\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+i0)^2}\times\not{k}\,\frac{1}{\not{k}+\not{p}-m_e+i0}\;\not{k}.\tag{S.16}$$

Now, let us simplify the fermionic factors here. Using eq. (3) twice, we have

$$\begin{aligned}
\not{k}\times\frac{1}{\not{k}+\not{p}-m_e+i0}\times\not{k}\;&=\;\left(1\;-\;(\not{p}-m)\times\frac{1}{\not{k}+\not{p}-m+i0}\right)\times\not{k}\\
&=\not{k}\;-\;(\not{p}-m)\times\left(1\;-\;\frac{1}{\not{k}+\not{p}-m+i0}\times\not{k}\right)\\
&=\not{k}\;-\;(\not{p}-m)\;+\;(\not{p}-m)\times\frac{1}{\not{k}+\not{p}-m+i0}\times(\not{p}-m),
\end{aligned}\tag{S.17}$$

and therefore

$$\int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+i0)^2} \times \not{k}\,\frac{1}{\not{k}+\not{p}-m_e+i0}\,\not{k} \;=$$

$$= \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{\not{k}}{(k^2+i0)^2} \;-\; (\not{p}-m)\times \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+i0)^2} \qquad \text{(S.18)}$$

$$+\; (\not{p}-m)\times \left(\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+i0)^2}\,\frac{1}{\not{k}+\not{p}-m+i0}\right)\times(\not{p}-m).$$

Note that first integral on the right hand side vanishes by the $k \to -k$ symmetry. The second integral does not vanish, but it's a $\not{p}$–independent constant multiplied by the $(\not{p}-m)$ factor. Finally, the third integral does depend on $\not{p}$, but it's multiplied by two $(\not{p}-m)$ factors. Thus, putting it all together, we find

$$\Sigma^{\text{1 loop}}_{\xi}(\not{p}) \;=\; \Sigma^{\text{1 loop}}_{F}(\not{p}) \;-\; (\not{p}-m)\times e^2(\xi-1)\int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2} \;+\; O\big((\not{p}-m)^2\big). \qquad (5)$$

Quod Ergo Demonstratum.

Problem **1**(d):

Eq. (5) does not include the counterterms δ_m and δ_2 which modify the propagator correction according to

$$\Sigma^{\text{net}}(\not{p}) \;=\; \Sigma^{\text{loops}}(\not{p}) \;+\; \Delta m \;-\; \delta_2\times\not{p}. \qquad \text{(S.19)}$$

The counterterms cancel out the loop corrections when the external momentum is on-shell, thus

$$\delta_2 \;=\; \frac{d\Sigma^{\text{loops}}}{d\,\not{p}}\bigg|_{\not{p}=m} \qquad \text{(S.20)}$$

and

$$\delta_m \;-\; m\delta_2 \;=\; -\,\Sigma^{\text{loops}}\Big|_{\not{p}=m}\;. \qquad \text{(S.21)}$$

Hence, eq. (5) implies that at the one-loop level

$$\delta_2^{\text{1 loop}}(\xi) \;=\; \delta_2^{\text{1 loop}}(\text{Feynman}) \;-\; (\xi-1)\times e^2 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2} \qquad (6)$$

while the $\delta_m - m\delta - 2$ counterterm combination is gauge invariant.

Plugging the gauge-dependent counterterms back into eq. (S.19), we find that the net propagator correction behaves as

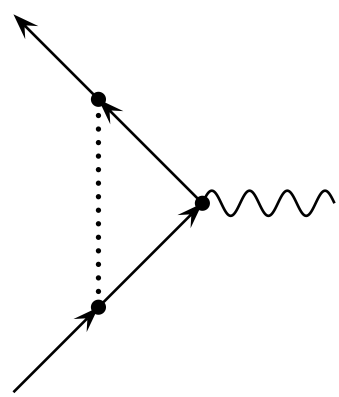
$$\Sigma_{\xi}^{\text{net}}(\not{p}) \;=\; \Sigma_{F}^{\text{net}}(\not{p}) \;+\; O\big((\not{p}-m)^2\big). \tag{S.22}$$

Thus, unlike the net vertex correction we considered in parts (a–b), the propagator correction does depend on the gauge fixing of the EM field. This is a special case of a general rule: The physical *on-shell* quantities (such as the on-shell form factors) must be gauge invariant, but the off-shell amplitudes may be gauge-dependent.

Going back to the counterterms, in the previous homework set we verified that $\delta_1=\delta_2$ in the Feynman gauge. In this set, we see that according to eqs. (4) and (6), the two counterterms have exactly similar dependence on the gauge-fixing parameter ξ. Therefore, even though the counterterms are gauge-dependent, the Ward identity holds in any gauge: $\forall\xi$, $\delta_1=\delta_2$.

Problem **2**(a):

The scalar field affects the muon's QED vertex through loop diagrams containing the scalar's propagators. At the one-loop level, there is one such diagram



(S.23)

which contributes

$$\begin{aligned}\Delta^{(\Phi)}[ie\Gamma^\mu(p',p)] \;&=\; \int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-M_s^2+i0}\times(-ig)\frac{i}{\not{p}'+\not{k}-m+i0}(ie\gamma^\mu)\frac{i}{\not{p}+\not{k}-m+i0}(-ig)\\ &=\; -eg^2\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{\mathcal{D}}\end{aligned} \tag{S.24}$$

where the numerator is

$$\mathcal{N}^\mu \;=\; (\not{p}' + \not{k} + m)\,\gamma^\mu\,(\not{p} + \not{k} + m) \tag{S.25}$$

and the denominator is

$$\begin{aligned}\frac{1}{\mathcal{D}} &= \frac{1}{k^2 - M_s^2 + i0} \times \frac{1}{(k+p')^2 - m^2 + i0} \times \frac{1}{(k+p)^2 - m^2 + i0} \\ &= \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{(\ell^2 - \Delta + i0)^3}\,.\end{aligned} \tag{S.26}$$

As usual, in the Feynman parameter integral

$$\ell^2 \;-\; \Delta \;=\; z(k^2 - M_s^2) \;+\; x((k+p)^2 - m^2) \;+\; y((k+p')^2 - m^2) \tag{S.27}$$

and hence

$$\ell \;=\; k \;+\; xp \;+\; yp', \tag{S.28}$$

$$\Delta \;=\; zM_s^2 \;+\; (1-z)^2 m^2 \;-\; xyq^2. \tag{S.29}$$

Altogether, we now have

$$\Delta^{(\Phi)}\Gamma^\mu(p',p) \;=\; 2ig^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{(\ell^2 - \Delta + i0)^3}\,, \tag{S.30}$$

and our next task is to simplify the numerator (S.25) in the present context. That is, we re-express $\mathcal{N}^\mu$ in terms of the shifted loop momentum ℓ, discard terms which integrate to zero (because they are odd with respect to $\ell \to -\ell$ or $x \leftrightarrow y$ symmetries), and also make use of the $\bar{u}(p')\,\Gamma^\mu\,u(p)$ context which allows us to substitute $\not{p} \to m$ in the rightmost factor and $\not{p}' \to m$ in the leftmost factor. Thus, proceeding similarly to QED vertex correction (*cf.* the supplementary note), we obtain

$$\begin{aligned}
\mathcal{N}^\mu \;&=\; \big((\not\ell - x\not p - y\not p') + \not p' + m\big)\,\gamma^\mu\,\big((\not\ell - x\not p - y\not p') + \not p + m\big)\\
&\cong \not\ell\gamma^\mu\not\ell \;+\; (z\not p' + x\not q + m)\,\gamma^\mu\,(z\not p - y\not q + m)\\
&\cong\; \gamma^\lambda\gamma^\mu\gamma^\nu \times \frac{\ell^2 g_{\lambda\nu}}{D} \;+\; ((z+1)m + x\not q)\,\gamma^\mu\,((z+1)m - y\not q)\\
&=\; \frac{2-D}{D}\,\ell^2\,\gamma^\mu \;+\; (z+1)^2m^2\,\gamma^\mu \;-\; xy\not q\gamma^\mu\not q \;+\; (z+1)m\Big((x-y)q^\mu \;+\; (x+y)i\sigma^{\mu\nu}q_\nu\Big)\\
&\cong\; \Big(-\frac{D-2}{D}\,\ell^2 \;+\; (1+z)^2m^2 \;+\; xyq^2\Big)\times\gamma^\mu \;+\; 2(1-z^2)m^2\times\frac{i\sigma^{\mu\nu}q_\nu}{2m}\,.
\end{aligned} \tag{S.31}$$

The Dirac-matrix structure of the last line here allows us to re-write eq. (S.30) in terms of the muons' form factors $F_1(q^2)$ and $F_2(q^2)$ as

$$\Delta^{(\Phi)}F_1(q^2) \;=\; 2ig^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{-\frac{D-2}{D}\ell^2 \;+\; (1+z)^2m^2 \;+\; xyq^2}{(\ell^2-\Delta+i0)^3}$$
$$+\;\Delta^{(\Phi)}\delta_1\,, \tag{S.32}$$

$$\Delta^{(\Phi)}F_2(q^2) \;=\; 2ig^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{2m^2(1-z^2)}{(\ell^2-\Delta+i0)^3}\,, \tag{S.33}$$

where the counterterm correction on the second line of eq. (S.32) follows from the value of the integral on the top line for $q^2=0$.

Fortunately, for the purpose of this problem we need only the second form factor $F_2(q^2)$, so we don't need to worry about the counterterm. Also, the momentum integral in eq. (S.33) is much simpler because the numerator is ℓ independent, and the denominator makes for a convergent integral

$$\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i0)^3} \;=\; \frac{-i}{32\pi^2\,\Delta}\,. \tag{S.34}$$

Therefore,

$$\Delta^{(\Phi)}F_2(q^2) \;=\; \frac{g^2}{16\pi^2}\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z^2)}{\Delta} \tag{S.35}$$

where Δ is as in eq. (S.29), and hence

$$\begin{aligned}\Delta^{(\Phi)} a_\mu &= \Delta^{(\Phi)} F_2(q^2 = 0) \\ &= \frac{g^2}{16\pi^2} \iiint_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\, \frac{2m^2(1-z^2)}{zM_s^2 + (1-z)^2 m^2} \\ &= \frac{g^2}{16\pi^2} \int_0^1 dz\, (1-z) \times \frac{2m^2(1-z^2)}{zM_s^2 + (1-z)^2 m^2}\,.\end{aligned} \tag{S.36}$$

The last integral here is a complicated function of the muon-to-scalar mass ratio m/M_s, but for the problem at hand, the scalar is much heavier than the muon. Hence, we approximate the denominator according to

$$\begin{aligned} zM_s^2 \;+\; (1-z)^2 m^2 \;&\approx \begin{cases} zM_s^2 \;+\; 0 & \text{except when } z \approx 0 \\ zM_s^2 \;+\; m^2 & \text{for } z \approx 0 \end{cases} \\ &\approx\; zM_s^2 \;+\; m^2\; \forall z, \end{aligned} \tag{S.37}$$

and consequently evaluate

$$\int_0^1 dz\, \frac{2m^2(1-z^2)(1-z)}{zM_s^2 + m^2} \;=\; 2\frac{m^2}{M_s^2} \left(\log\frac{M_s^2}{m^2} \;-\; \frac{7}{6} \;+\; O\left(\frac{m^2}{M^2}\right)\right). \tag{S.38}$$

Thus, to the leading orders in the Yukawa coupling g and in the m/M_s mass ration, the scalar's effect on the anomalous magnetic moment of the muon amounts to

$$\Delta^{(\Phi)} a_\mu \;\approx\; \frac{g^2}{8\pi^2} \frac{m^2}{M_s^2} \left(\log\frac{M_s^2}{m^2} \;-\; \frac{7}{6}\right). \tag{S.39}$$

Experimentally, we know muon's anomalous magnetic moment agrees with the minimal Standard model to a very high accuracy. If any extra fields are present, their effect on the a_μ should not be significantly stronger than the experimental error $\Delta^{\exp} a_\mu \approx 80 \cdot 10^{-11}$ —

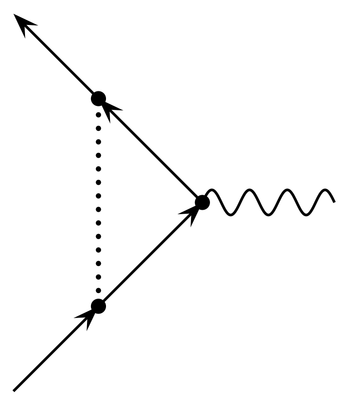
otherwise, we would have noticed the discrepancy! Hence, eq. (S.39) provides us with an upper limit on the Yukawa coupling of an additional scalar field, namely

$$\frac{g^2}{8\pi^2}\frac{m^2}{M_s^2}\left(\log\frac{M_s^2}{m^2} - \frac{7}{6}\right) \lesssim 80\cdot 10^{-11}. \tag{S.40}$$

For $M_s \approx 200$ GeV $\approx 2000\,m_\mu$, we must have $g^2 \lesssim 0.018$, but for a heavier scalar we would have a weaker limit.

Problem **2**(b):

At the one-loop level, the axion's effect on the QED vertex of the muon follows from a single diagram



(S.41)

which looks exactly like (S.23) but evaluates differently because of different Yukawa vertices:

$$\frac{2m_\mu}{f_a}\gamma^5 \equiv g\gamma^5 \tag{S.42}$$

instead of $-ig$. Also, the axion is lighter than the muon, $M_a \ll m_\mu$.

Consequently,

$$\begin{aligned}\Delta^{(a)}[ie\Gamma^\mu(p',p)] &= \int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-M_a^2+i0}\times(g\gamma^5)\frac{i}{\not{p}'+\not{k}-m+i0}(ie\gamma^\mu)\frac{i}{\not{p}+\not{k}-m+i0}(g\gamma^5)\\ &= -2eg^2\iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\limits_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\frac{\mathcal{N}^\mu}{(\ell^2-\Delta+i0)^3}\end{aligned} \tag{S.43}$$

where the denominator is exactly as in part (a) of the problems (*cf.* eqs. (S.28) and (S.29)) except for $M_s^2 \to M_a^2$, but the numerator is now

$$\begin{aligned} \mathcal{N}^\mu &= -\gamma^5 \times (\not{k} + \not{p}' + m) \times \gamma^\mu \times (\not{k} + \not{p} + m) \times \gamma^5 \\ &= +(\not{k} + \not{p}' - m) \times \gamma^\mu \times (\not{k} + \not{p} - m). \end{aligned} \tag{S.44}$$

As in part (a), we need to re-express this numerator in terms of the shifted loop momentum ℓ and then discard terms which integrate to zero or vanish on-shell (in the context of $\bar{u}'\Gamma^\mu u$). Proceeding similarly to eq. (S.31), we obtain

$$\mathcal{N}^\mu \cong \left(-\frac{D-2}{D}\,\ell^2 + (1-z)^2 m^2 + xyq^2\right) \times \gamma^\mu - 2(1-z)^2 m^2 \times \frac{i\sigma^{\mu\nu} q_\nu}{2m} \tag{S.45}$$

and consequently

$$\begin{aligned} \Delta^{(a)} F_1(q^2) &= 2ig^2 \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int\limits_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\, \frac{-\frac{D-2}{D}\ell^2 + (1-z)^2 m^2 + xyq^2}{(\ell^2 - \Delta + i0)^3} \\ &\quad + \Delta^{(\Phi)}\delta_1\,, \end{aligned} \tag{S.46}$$

$$\Delta^{(a)} F_2(q^2) = 2ig^2 \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int\limits_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\, \frac{-2m^2(1-z)^2}{(\ell^2 - \Delta + i0)^3}\,, \tag{S.47}$$

where the counterterm correction on the second line of eq. (S.46) follows from the value of the integral on the first line for $q^2 = 0$.

Again, we focus on the F_2 form factor. Evaluating the momentum integral according to eq. (S.34), we have

$$\Delta^{(a)} F_2(q^2) = \frac{g^2}{16\pi^2} \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\, \frac{-2m^2(1-z)^2}{\Delta} \tag{S.48}$$

and hence

$$\begin{aligned} \Delta^{(a)} a_\mu &= \Delta^{(a)} F_2(q^2 = 0) \\ &= \frac{g^2}{16\pi^2} \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\, \frac{-2m^2(1-z)^2}{zM_a^2 + (1-z)^2 m^2} \\ &= \frac{g^2}{16\pi^2} \int\limits_0^1 dz\,(1-z) \times \frac{-2m^2(1-z)^2}{zM_a^2 + (1-z)^2 m^2}\,. \end{aligned} \tag{S.49}$$

Unlike the scalar field in part (a) of the problem, the axion is light compared to the muon. Consequently, we approximate

$$\int_0^1 dz\,(1-z)\times\frac{-2m^2(1-z)^2}{zM_a^2+(1-z)^2m^2}\approx\int_0^1 dz\,(1-z)\times\frac{-2m^2(1-z)^2}{(1-z)^2m^2}\;=\;-1 \tag{S.50}$$

and hence

$$\Delta^{(a)}a_\mu\;\approx\;-\frac{g^2}{16\pi^2}\;=\;-\frac{m_\mu^2}{4\pi^2 f_a^2}\,. \tag{S.51}$$

Again, given the experimental data on the muon's magnetic moment, eq. (S.51) imposes an upper limit on the strength of the axion's coupling, or equivalently a lower limit on the axion's scale parameter $f_a \gtrsim 600$ GeV.