Problem **2**(a):

First a little lemma: For a unitary $U(x)$,

$$\begin{aligned} \partial_\mu\Big(U(x)\Phi(x)U^\dagger(x)\Big) \;&=\; (\partial_\mu U)\Phi U^\dagger \;+\; U(\partial_\mu\Phi)U^\dagger \;+\; U\Phi\Big(\partial_\mu U^\dagger \;=\; -U^\dagger(\partial_\mu U)U^\dagger)\Big) \\ &=\; U(\partial_\mu\Phi)U^\dagger + \big[(\partial_\mu U)U^\dagger, U\Phi U^\dagger\big] \end{aligned} \tag{S.1}$$

And now let's combine gauge transformation laws (2) and

$$A'_\mu(x) \;=\; U(x)A_\mu(x)U^\dagger(x) \;+\; i(\partial_\mu U(x))U^\dagger(x), \tag{S.2}$$

and apply them to the derivative (3):

$$\begin{aligned} D'_\mu\Phi'(x) \;&=\; \partial_\mu\Phi'(x) \;+\; i\big[A'_\mu(x),\Phi'(x)\big] \\ &=\; \partial_\mu(U\Phi U^\dagger) \;+\; i\big[UA_\mu U^\dagger, U\Phi U^\dagger\big] \;-\; \big[(\partial_\mu U)U^\dagger, U\Phi U^\dagger\big] \\ &=\; U(\partial_\mu\Phi)U^\dagger + \big[(\partial_\mu U)U^\dagger, U\Phi U^\dagger\big] \;+\; iU[A_\mu,\Phi]U^\dagger \;-\; \big[(\partial_\mu U)U^\dagger, U\Phi U^\dagger\big] \\ &=\; U\big(\partial_\mu\Phi \;+\; i[A_\mu,\Phi]\big)U^\dagger \\ &\equiv\; U(x)\big(D_\mu\Phi(x)\big)U^\dagger(x). \end{aligned} \tag{S.3}$$

In other words, the covariant derivative defined by eq. (3) is indeed covariant. $\mathcal{Q.E.D.}$

Problem **2**(b):

$$\begin{aligned} D_\mu D_\nu\Phi \;&=\; D_\mu(\partial_\nu\Phi \;+\; i[A_\nu,\Phi]) \;=\; \partial_\mu\big(\partial_\nu\Phi \;+\; i[A_\nu,\Phi]\big) \;+\; i\big[A_\mu,(\partial_\nu\Phi \;+\; i[A_\nu,\Phi])\big] \\ &=\; \partial_\mu\partial_\nu\Phi \;+\; i[(\partial_\mu A_\nu),\Phi] \;+\; i[A_\nu,\partial_\mu\Phi] \;+\; i[A_\mu,\partial_\nu\Phi] \;-\; [A_\mu,[A_\nu,\Phi]]. \end{aligned} \tag{S.4}$$

Similarly,

$$D_\nu D_\mu\Phi \;=\; \partial_\nu\partial_\mu\Phi \;+\; i[(\partial_\nu A_\mu),\Phi] \;+\; i[A_\mu,\partial_\nu\Phi] \;+\; i[A_\nu,\partial_\mu\Phi] \;-\; [A_\nu,[A_\mu,\Phi]]. \tag{S.5}$$

Three out of five terms on the right hand sides of these formulæ are identical and hence cancel out of the difference $D_\mu D_\nu\Phi - D_\nu D_\mu\Phi$. The remaining terms comprise the commutator

$$\begin{aligned}
[D_\mu, D_\nu]\Phi \;&=\; i[(\partial_\mu A_\nu),\Phi] \;-\; i[(\partial_\nu A_\mu),\Phi] \;-\; [A_\mu,[A_\nu,\Phi]] \;+\; [A_\nu,[A_\mu,\Phi]] \\
&=\; i[(\partial_\mu A_\nu - \partial_\nu A_\mu),\Phi] \;-\; [[A_\mu,A_\nu],\Phi] \\
&\equiv\; i[F_{\mu\nu},\Phi].
\end{aligned} \tag{S.6}$$

Problem $\mathbf{2}$(c):

First, let us evaluate

$$\begin{aligned}
D_\lambda F_{\mu\nu} \;&=\; \partial_\lambda\bigl(\partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; i[A_\mu,A_\nu]\bigr) \;+\; i\bigl[A_\lambda,(\partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; i[A_\mu,A_\nu])\bigr] \\
&=\; \Bigl(\partial_\lambda\partial_\mu A_\nu \;-\; \partial_\lambda\partial_\nu A_\mu\Bigr) \;+\; i\Bigl([A_\mu,\partial_\lambda A_\nu] \;-\; [A_\nu,\partial_\lambda A_\mu]\Bigr) \\
&\quad +\; i\Bigl([A_\lambda,\partial_\mu A_\nu] \;-\; [A_\lambda,\partial_\nu A_\mu]\Bigr) \;-\; [A_\lambda,[A_\mu,A_\nu]].
\end{aligned} \tag{S.7}$$

For each group of terms here, summing over cyclic permutation of the Lorentz indices $\lambda \to \mu \to \nu \to \lambda$ produces a zero:

$$\begin{aligned}
(\partial_\lambda\partial_\mu A_\nu - \partial_\lambda\partial_\nu A_\mu) \,+\, (\partial_\mu\partial_\nu A_\lambda - \partial_\mu\partial_\lambda A_\nu) \,+\, (\partial_\nu\partial_\lambda A_\mu - \partial_\nu\partial_\mu A_\lambda) \;&=\; 0, \\
\bigl([A_\mu,\partial_\lambda A_\nu] - [A_\nu,\partial_\lambda A_\mu]\bigr) \,+\, \bigl([A_\nu,\partial_\mu A_\lambda] - [A_\lambda,\partial_\mu A_\nu]\bigr) \,+\, \bigl([A_\lambda,\partial_\nu A_\mu] - [A_\mu,\partial_\nu A_\lambda]\bigr) \;&=\; 0, \\
\bigl([A_\lambda,\partial_\mu A_\nu] - [A_\lambda,\partial_\nu A_\mu]\bigr) \,+\, \bigl([A_\mu,\partial_\nu A_\lambda] - [A_\mu,\partial_\lambda A_\nu]\bigr) \,+\, \bigl([A_\nu,\partial_\lambda A_\mu] - [A_\nu,\partial_\mu A_\lambda]\bigr) \;&=\; 0, \\
[A_\lambda,[A_\mu,A_\nu]] \;+\; [A_\mu,[A_\nu,A_\lambda]] \;+\; [A_\nu,[A_\lambda,A_\mu]] \;&=\; 0,
\end{aligned} \tag{S.8}$$

and consequently

$$D_\lambda F_{\mu\nu} \;+\; D_\mu F_{\nu\lambda} \;+\; D_\nu F_{\lambda\mu} \;=\; 0. \tag{9}$$

Problem $\mathbf{2}$(d):

$$\begin{aligned}
\delta\left(F_{\mu\nu} \;=\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; i[A_\mu,A_\nu]\right) \;&=\; \partial_\mu\delta A_\nu \;-\; \partial_\nu\delta A_\mu \;+\; i[\delta A_\mu,A_\nu] \;+\; i[A_\mu,\delta A_\nu] \\
&=\; D_\mu\delta A_\nu \;-\; D_\nu\delta A_\mu
\end{aligned} \tag{S.9}$$

Problem $\mathbf{2}$(e): Classical equations of motion for the fermionic fields are easy: The Lagrangian (4) contains spacetime derivatives of the $\Psi(x)$ field but not of its conjugate $\overline{\Psi}(x)$, hence

$$\frac{\partial\mathcal{L}}{\partial\overline{\Psi}} \equiv (i\gamma^\mu D_\mu - m)\Psi(x) \;=\; 0. \tag{S.10}$$

By conjugation,

$$\overline{\Psi}(-i\gamma^\mu \overleftarrow{D}_\mu - m) \;\equiv\; -i\partial_\mu\overline{\Psi}\gamma^\mu \;-\; \overline{\Psi}\gamma^\mu A_\mu \;-\; m\overline{\Psi} \;=\; 0. \tag{S.11}$$

For the bosonic field $A_\mu(x) = \sum_a A^a_\mu(x)\frac{\lambda^a}{2}$ we have the Euler–Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial A^a_\mu} \;-\; \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu A^a_\mu)} \;=\; 0, \tag{S.12}$$

but the exact evaluation of the derivatives here is painful, and it's easier to directly evaluate the Lagrangian variation $\delta\mathcal{L}$. In light of the previous question (d), we have

$$\begin{aligned}
\delta\,\mathrm{tr}\Big(F^{\mu\nu}F_{\mu\nu}\Big) \;&=\; 2\,\mathrm{tr}\Big(F^{\mu\nu}\,\delta F_{\mu\nu}\Big) \;=\; 2\,\mathrm{tr}\Big(F^{\mu\nu}(D_\mu\delta A_\nu - D_\nu\delta A_\mu)\Big)\\
&=\; 4\,\mathrm{tr}\Big(F^{\mu\nu}\,D_\mu\delta A_\nu\Big) \;=\; 4\partial_\mu\,\mathrm{tr}\Big(F^{\mu\nu}\,\delta A_\nu\Big) \;-\; 4\,\mathrm{tr}\Big(\delta A_\nu\times D_\mu F^{\mu\nu}\Big)\\
&=\; \text{total derivative} \;-\; 2\sum_a \delta A^a_\nu\times(D_\mu F^{\mu\nu})^a
\end{aligned} \tag{S.13}$$

where the last equality follows from

$$\delta A_\nu = \sum_a \delta A^a_\mu\,\frac{\lambda^a}{2}, \quad D_\mu D^{\mu\nu} = \sum_b (D_\mu F^{\mu\nu})^b\,\frac{\lambda^b}{2}, \quad \text{and} \quad \mathrm{tr}\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right) = \frac{\delta^{ab}}{2}. \tag{S.14}$$

Now consider the fermionic Lagrangian which depends on the gauge fields according to

$$\overline{\Psi}(i\gamma^\mu D_\mu - m)\Psi \;=\; \overline{\Psi}\left(i\gamma^\mu\partial_\mu \;-\; \gamma^\mu\sum_a A^a_\mu\,\frac{\lambda^a}{2} \;-\; m\right)\Psi. \tag{S.15}$$

Consequently

$$\delta\left(\overline{\Psi}(i\gamma^\mu D_\mu - m)\Psi\right) \;=\; -\sum_a \delta A_\nu\times\overline{\Psi}\gamma^\nu\frac{\lambda^a}{2}\Psi, \tag{S.16}$$

and hence

$$\delta\mathcal{L} \;=\; \sum_a \delta A_\nu \times \left(g^2 (D_\mu F^{\mu\nu})^a \;-\; \overline{\Psi}\gamma^\nu \frac{\lambda^a}{2}\Psi \right) \;+\; \text{a total derivative,} \tag{S.17}$$

or equivalently

$$\delta S \;=\; \sum A \int d^4x\, \delta A_\nu(x) \times \left(g^{-2} 2 (D_\mu F^{\mu\nu}(x))^a \;-\; \overline{\Psi}(x)\gamma^\nu \frac{\lambda^a}{2}\Psi(x) \right). \tag{S.18}$$

Therefore, the classical gauge fields $A^a_\mu(x)$ satisfy the Yang–Mills equations

$$(D_\mu F^{\mu\nu}(x))^a \;=\; g^2 \overline{\Psi}(x)\gamma^\nu \frac{\lambda^a}{2}\Psi(x) \tag{S.19}$$

Note that eqs. (S.19) apply to the vector fields normalized by the local symmetry transformations. The canonically normalized vector fields

$$\mathbf{A}^a_\mu \;=\; \frac{1}{g} A^a_\mu \quad \text{and} \quad \mathbf{F}^a_{\mu\nu} \;=\; \frac{1}{g} F^a_{\mu\nu} \;=\; \partial_\mu \mathbf{A}^a_\nu \;-\; \partial_\nu \mathbf{A}^a_\mu \;-\; g f^{abc} \mathbf{A}^b_\mu \mathbf{A}^c_\nu \tag{S.20}$$

satisfy

$$(D_\mu \mathbf{F}^{\mu\nu})^a \;\equiv\; \partial_\mu \mathbf{F}^a_{\mu\nu} \;-\; g f^{abc} \mathbf{A}^b_\mu \mathbf{F}^{c\mu\nu} \;=\; g J^{a\nu} \equiv\; g\overline{\Psi}(x)\gamma^\nu \frac{\lambda^a}{2}\Psi(x). \tag{S.21}$$

<u>Addendum to problem **2**</u>: This was not assigned, but perhaps should have been.

Note that the Yang–Mills equations (S.19) or (S.21) require the fermionic current

$$J^a_\nu(x) \;=\; \overline{\Psi}(x)\gamma_\nu \frac{\lambda^a}{2}\Psi(x) \tag{S.22}$$

to be *covariantly conserved*:

$$D_\nu J^\nu \;=\; 0, \quad i.e., \quad \partial_\nu J^{a\nu} \;-\; g f^{abc} A^b_\nu J^{\nu c} \;=\; 0. \tag{S.23}$$

Indeed,

$$D_\nu J^\nu \;\propto\; D_\nu (D_\mu F^{\mu\nu}) \;=\; -\tfrac{1}{2}[D_\mu, D_\nu] F^{\mu\nu} \;=\; -\tfrac{i}{2}[F_{\mu\nu}, F^{\mu\nu}] \;=\; 0. \tag{S.24}$$

where the first equality follows from $F^{\mu\nu} = -F^{\nu\mu}$, the second from question (b), and the third from the fact that $F^{\mu\nu}$ commutes with itself.

As usual, current-conservation equations like (S.23) can be derived from the fermionic field equations (S.10) and (S.11):

$$\begin{aligned}
\partial_\nu J^{a\nu} &= (\partial_\nu \overline{\Psi}\gamma^\nu)\frac{\lambda^a}{2}\Psi \;+\; \overline{\Psi}\frac{\lambda^a}{2}(\gamma^\nu\partial_\nu\Psi) \\
&= i\overline{\Psi}\left(m \,+\, \gamma^\nu A^b_\nu \frac{\lambda^b}{2}\right) \times \frac{\lambda^a}{2}\Psi \;+\; \overline{\Psi}\frac{\lambda^a}{2} \times -i\left(m \,+\, \gamma^\nu A^b_\nu \frac{\lambda^b}{2}\right)\Psi \\
&= iA^b_\nu \times \overline{\Psi}\gamma^\nu\left[\frac{\lambda^b}{2}, \frac{\lambda^a}{2}\right]\Psi \;=\; iA^b_\nu \times \overline{\Psi}\gamma^\nu i f^{bac}\frac{\lambda^c}{2}\Psi \\
&= +f^{abc}A^b_\nu J^{c\nu} \;\equiv\; g f^{abc}\mathbf{A}^b_\nu J^{c\nu}
\end{aligned} \tag{S.25}$$

and therefore

$$D_\nu J^{a\nu} \;=\; \partial_\nu J^{a\nu} \;-\; g f^{abc}\mathbf{A}^b_\nu J^{c\nu} \;=\; 0. \tag{S.26}$$

Note however that a *covariantly conserved* local current does not lead to a conserved global charge. Indeed, unlike QED the non-abelian gauge theories do not have conserved charges, and this leads to all kinds of complications.