Problem 2(a):
First a little lemma: For a unitary $U(x)$,

\[
\partial_\mu \left( U(x)\Phi(x)U^\dagger(x) \right) = (\partial_\mu U)\Phi U^\dagger + U(\partial_\mu \Phi)U^\dagger + U\Phi \left( \partial_\mu U^\dagger = -U^\dagger (\partial_\mu U)U^\dagger \right)
\]

\[
= U(\partial_\mu \Phi)U^\dagger + [(\partial_\mu U)U^\dagger, U\Phi U^\dagger]
\]

(S.1)

And now let’s combine gauge transformation laws (2) and

\[
A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + i(\partial_\mu U(x))U^\dagger(x),
\]

(S.2)

and apply them to the derivative (3):

\[
D'_\mu \Phi'(x) = \partial_\mu \Phi'(x) + i[A'_\mu(x), \Phi'(x)]
\]

\[
= \partial_\mu (U\Phi U^\dagger) + i[UA_\mu U^\dagger, U\Phi U^\dagger] - [(\partial_\mu U)U^\dagger, U\Phi U^\dagger]
\]

\[
= U(\partial_\mu \Phi)U^\dagger + [(\partial_\mu U)U^\dagger, U\Phi U^\dagger] + iU[A_\mu, \Phi]U^\dagger - [(\partial_\mu U)U^\dagger, U\Phi U^\dagger]
\]

(S.3)

\[
\equiv U(x)(D_\mu \Phi(x))U^\dagger(x).
\]

In other words, the covariant derivative defined by eq. (3) is indeed covariant. \textit{Q.E.D.}

Problem 2(b):

\[
D_\mu D_\nu \Phi = D_\mu (\partial_\nu \Phi + i[A_\nu, \Phi]) = \partial_\mu (\partial_\nu \Phi + i[A_\nu, \Phi]) + i[A_\mu, (\partial_\nu \Phi + i[A_\nu, \Phi])]
\]

\[
= \partial_\mu \partial_\nu \Phi + i[(\partial_\mu A_\nu), \Phi] + i[A_\nu, \partial_\mu \Phi] + i[A_\mu, \partial_\nu \Phi] - [A_\mu, [A_\nu, \Phi]]
\]

(S.4)

Similarly,

\[
D_\nu D_\mu \Phi = \partial_\nu \partial_\mu \Phi + i[(\partial_\nu A_\mu), \Phi] + i[A_\mu, \partial_\nu \Phi] + i[A_\nu, \partial_\mu \Phi] - [A_\nu, [A_\mu, \Phi]].
\]

(S.5)

Three out of five terms on the right hand sides of these formulae are identical and hence cancel.
out of the difference $D_\mu D_\nu \Phi - D_\nu D_\mu \Phi$. The remaining terms comprise the commutator

$$[D_\mu, D_\nu] \Phi = i[(\partial_\mu A_\nu), \Phi] - i[(\partial_\nu A_\mu), \Phi] - [A_\mu, [A_\nu, \Phi]] + [A_\nu, [A_\mu, \Phi]]$$

$$= i[(\partial_\mu A_\nu - \partial_\nu A_\mu), \Phi] - [[A_\mu, A_\nu], \Phi]$$

$$\equiv i[F_{\mu\nu}, \Phi].$$

**Problem 2(c):**

First, let us evaluate

$$D_\lambda F_{\mu\nu} = \partial_\lambda (\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]) + i[A_\lambda, (\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu])]$$

$$= \left( \partial_\lambda \partial_\mu A_\nu - \partial_\lambda \partial_\nu A_\mu \right) + i \left( [A_\mu, \partial_\lambda A_\nu] - [A_\nu, \partial_\lambda A_\mu] \right)$$

$$+ i \left( [A_\lambda, \partial_\mu A_\nu] - [A_\nu, \partial_\mu A_\lambda] \right) - [A_\lambda, [A_\mu, A_\nu]].$$

For each group of terms here, summing over cyclic permutation of the Lorentz indices $\lambda \to \mu \to \nu \to \lambda$ produces a zero:

$$(\partial_\lambda \partial_\mu A_\nu - \partial_\lambda \partial_\nu A_\mu) + (\partial_\nu \partial_\lambda A_\mu - \partial_\mu \partial_\lambda A_\nu) + (\partial_\mu \partial_\lambda A_\nu - \partial_\nu \partial_\lambda A_\mu) = 0,$$

$$([A_\mu, \partial_\nu A_\lambda] - [A_\nu, \partial_\lambda A_\mu]) + ([A_\nu, \partial_\mu A_\lambda] - [A_\lambda, \partial_\mu A_\nu]) + ([A_\lambda, \partial_\mu A_\nu] - [A_\mu, \partial_\nu A_\lambda]) = 0,$$

$$([A_\lambda, \partial_\mu A_\nu] - [A_\nu, \partial_\mu A_\lambda]) + ([A_\nu, \partial_\lambda A_\mu] - [A_\mu, \partial_\lambda A_\nu]) + ([A_\lambda, \partial_\nu A_\mu] - [A_\mu, \partial_\nu A_\lambda]) = 0,$$

$$\left( [A_\lambda, [A_\mu, A_\nu]] + [A_\mu, [A_\nu, A_\lambda]] + [A_\nu, [A_\lambda, A_\mu]] \right) = 0,$$

(S.8)

and consequently

$$D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0. \quad (9)$$

**Problem 2(d):**

$$\delta (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]) = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu + i[\delta A_\mu, A_\nu] + i[A_\mu, \delta A_\nu]$$

$$= D_\mu \delta A_\nu - D_\nu \delta A_\mu$$

(S.9)

**Problem 2(e):** Classical equations of motion for the fermionic fields are easy: The Lagrangian (4) contains spacetime derivatives of the $\Psi(x)$ field but not of its conjugate $\bar{\Psi}(x),$
hence
\[ \frac{\partial \mathcal{L}}{\partial \Psi} \equiv (i\gamma^\mu D_\mu - m)\Psi(x) = 0. \] (S.10)

By conjugation,
\[ \overline{\Psi}(-i\gamma^\mu D_\mu - m) \equiv -i\partial_\mu \overline{\Psi}\gamma^\mu - \overline{\Psi}\gamma^\mu A_\mu - m\overline{\Psi} = 0. \] (S.11)

For the bosonic field \( A_\mu(x) = \sum_a A^a_\mu(x) \frac{\lambda^a}{2} \) we have the Euler–Lagrange equation
\[ \frac{\partial \mathcal{L}}{\partial A^a_\mu} - \partial_\nu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu A^a_\mu)} \right) = 0, \] (S.12)
but the exact evaluation of the derivatives here is painful, and it’s easier to directly evaluate the Lagrangian variation \( \delta \mathcal{L} \). In light of the previous question (d), we have
\[ \delta \left( F^{\mu\nu}F_{\mu\nu} \right) = 2 \text{tr} \left( F^{\mu\nu} \delta F_{\mu\nu} \right) = 2 \text{tr} \left( F^{\mu\nu} (D_\mu \delta A_\nu - D_\nu \delta A_\mu) \right) = 4 \text{tr} \left( F^{\mu\nu} D_\mu \delta A_\nu \right) = 4 \partial_\mu \text{tr} \left( F^{\mu\nu} \delta A_\nu \right) = 4 \text{tr} \left( \delta A_\nu \times D_\mu F^{\mu\nu} \right) \] (S.13)
where the last equality follows from
\[ \delta A_\nu = \sum_a \delta A^a_\mu \frac{\lambda^a}{2}, \quad D_\mu D^{\mu\nu} = \sum_b (D_\mu F^{\mu\nu})^b \frac{\lambda^b}{2}, \quad \text{and} \quad \text{tr} \left( \frac{\lambda^a \lambda^b}{2} \right) = \frac{\delta^{ab}}{2}. \] (S.14)

Now consider the fermionic Lagrangian which depends on the gauge fields according to
\[ \overline{\Psi}(i\gamma^\mu D_\mu - m)\Psi = \overline{\Psi} \left( i\gamma^\mu \partial_\mu - \gamma^\mu \sum_a A^a_\mu \frac{\lambda^a}{2} - m \right) \Psi. \] (S.15)
Consequently
\[ \delta \left( \overline{\Psi}(i\gamma^\mu D_\mu - m)\Psi \right) = -\sum_a \delta A_\nu \times \overline{\Psi}\gamma^\nu \frac{\lambda^a}{2} \Psi, \] (S.16)
and hence
\[ \delta \mathcal{L} = \sum_a \delta A_a \times \left( g^2 (D_\mu F^{\mu \nu})^a - \overline{\Psi} \gamma^\nu \frac{\lambda^a}{2} \Psi \right) + \text{a total derivative}, \quad (S.17) \]
or equivalently
\[ \delta S = \sum A \int d^4x \delta A_\nu (x) \times \left( g^{-2} 2 (D_\mu F^{\mu \nu} (x))^a - \overline{\Psi (x)} \gamma^\nu \frac{\lambda^a}{2} \Psi (x) \right). \quad (S.18) \]
Therefore, the classical gauge fields \( A_\mu^a (x) \) satisfy the Yang–Mills equations
\[ (D_\mu F^{\mu \nu} (x))^a = g^2 \overline{\Psi (x)} \gamma^\nu \frac{\lambda^a}{2} \Psi (x) \quad (S.19) \]

Note that eqs. (S.19) apply to the vector fields normalized by the local symmetry transformations. The canonically normalized vector fields
\[ A_\mu^a = \frac{1}{g} A_\mu^a \quad \text{and} \quad F_{\mu \nu}^a = \frac{1}{g} F_{\mu \nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c \quad (S.20) \]
satisfy
\[ (D_\mu F^{\mu \nu})^a = \partial_\mu F_{\mu \nu}^a - g f^{abc} A_\mu^b F_{\mu \nu}^c = g J_{\mu \nu}^a = \overline{\Psi (x)} \gamma^\nu \frac{\lambda^a}{2} \Psi (x). \quad (S.21) \]

Addendum to problem 2: This was not assigned, but perhaps should have been.

Note that the Yang–Mills equations (S.19) or (S.21) require the fermionic current
\[ J_\nu^a (x) = \overline{\Psi (x)} \gamma^\nu \frac{\lambda^a}{2} \Psi (x) \quad (S.22) \]
to be covariantly conserved:
\[ D_\nu J^\nu = 0, \quad \text{i.e.,} \quad \partial_\nu J_{\mu \nu}^a - g f^{abc} A_\nu^b J^c_{\mu \nu} = 0. \quad (S.23) \]
Indeed,
\[ D_\nu J^\nu \propto D_\nu (D_\mu F^{\mu \nu}) = -\frac{1}{2} [D_\mu, D_\nu] F^{\mu \nu} = -\frac{i}{2} [F_{\mu \nu}, F^{\mu \nu}] = 0. \quad (S.24) \]
where the first equality follows from \( F^{\mu \nu} = -F^{\nu \mu} \), the second from question (b), and the third from the fact that \( F^{\mu \nu} \) commutes with itself.
As usual, current-conservation equations like (S.23) can be derived from the fermionic field equations (S.10) and (S.11):

\[
\partial_\nu J^{\alpha\nu} = (\partial_\nu \overline{\Psi} \gamma^\nu) \frac{\lambda^\alpha}{2} \Psi + \overline{\Psi} \frac{\lambda^\alpha}{2} (\gamma^\nu \partial_\nu \Psi)
\]

\[
= i \overline{\Psi} \left( m + \gamma^\nu A^b_\nu \frac{\lambda^b}{2} \right) \times \frac{\lambda^\alpha}{2} \Psi + \overline{\Psi} \frac{\lambda^\alpha}{2} \times -i \left( m + \gamma^\nu A^b_\nu \frac{\lambda^b}{2} \right) \Psi
\]

\[
= i A^b_\nu \times \overline{\Psi} \gamma^\nu \left[ \frac{\lambda^b}{2} , \frac{\lambda^\alpha}{2} \right] \Psi = i A^b_\nu \times \overline{\Psi} \gamma^\nu i f^{abc} \frac{\lambda^c}{2} \Psi
\]

\[
= + f^{abc} A^b_\nu J^{c\nu} \equiv g f^{abc} A^b_\nu J^{c\nu}
\]

and therefore

\[
D_\nu J^{\alpha\nu} = \partial_\nu J^{\alpha\nu} - g f^{abc} A^b_\nu J^{c\nu} = 0. \quad \text{(S.26)}
\]

Note however that a covariantly conserved local current does not lead to a conserved global charge. Indeed, unlike QED the non-abelian gauge theories do not have conserved charges, and this leads to all kinds of complications.