

Problem 1(a):

Let us evaluate the trace of the Casimir operator \hat{C}_2 over the irreducible representation (r) . On one hand,

$$\begin{aligned} \text{tr}_{(r)} \left(\hat{C}_2 \stackrel{\text{def}}{=} \sum_a T^a T^a \right) &= \sum_a \text{tr}_{(r)} (T^a T^a) \\ \langle\langle \text{by eq. (1)} \rangle\rangle &= \sum_a R(r) \times (\delta^{aa} = 1) = R(r) \times \dim(G) \end{aligned} \quad (\text{S.1})$$

where $\dim(G) \stackrel{\text{def}}{=} \dim(\text{Adj}(G))$ is the number of the generators of the symmetry group G — which is also the dimension of the adjoint representation of G , hence the notation. On the other hand,

$$\text{tr}_{(r)} (\hat{C}_2) = \text{tr}_{(r)} \left(\hat{C}_2 \Big|_{(r)} \right) = \text{tr}_{(r)} (C(r) \times 1) = C(r) \times \dim(r). \quad (\text{S.2})$$

Together, eqs. (S.1) and (S.2) immediately imply eq. (2), *Q.E.D.*

For the special case of $G = SU(2)$, we have distinct irreducible representations uniquely specified by the value ‘isospin’ I and has $C(I) = I(I+1)$. Also, $\dim(I) = (2I+1)$ and $\dim(G) = 3$, hence,

$$R(I) = C(I) \times \frac{\dim(I)}{\dim(G)} = \frac{1}{3} I(I+1)(2I+1). \quad (\text{S.3})$$

Problem 1(b):

Unlike the Casimir value $C(r)$, the index $R(r)$ is well defined for any complete representation (r) , irreducible or otherwise. For a reducible representation

$$(r) = \bigoplus_{i=1}^n (r_i) \equiv (r_1) \oplus (r_2) \oplus \cdots \oplus (r_n)$$

one clearly has

$$\begin{aligned} \text{tr}_{(r)} (T^a T^b) &= \text{tr} \left(T^a T^b \Big|_{\bigoplus_{i=1}^n (r_i)} \right) = \sum_{i=1}^n \text{tr} \left(T^a T^b \Big|_{(r_i)} \right) \\ &= \sum_{i=1}^n \left(R(r_i) \times \delta^{ab} \right) = \delta^{ab} \times \sum_{i=1}^n R(r_i) \end{aligned} \quad (\text{S.4})$$

and thus

$$R(r) = \sum_{i=1}^n R(r_i). \quad (\text{S.5})$$

In particular, for a reducible representation

$$(r) = \bigoplus_{i=1}^n (I_i)$$

of the isospin group $SU(2)$, one has

$$R(r) = \sum_{i=1}^n \frac{1}{3} I(I+1)(2I+1). \quad (\text{S.6})$$

Now consider a bigger symmetry group G which contains the ‘isospin’ $SU(2)$ as a subgroup. Then any complete representation (r) of G is automatically a complete representation of the $SU(2) \subset G$. Generally, such (r) would be a reducible representation of the $SU(2)$ even if it were irreducible from the bigger group G point of view, thus we expect (r) to decompose into $(I_1) \oplus (I_2) \oplus \dots \oplus (I_n)$ from the $SU(2)$ point of view. (The isospins I_1, I_2, \dots, I_n may be all distinct or all equal or whatever.) Consequently, for $a, b = 1, 2, 3$, *i.e.* T^a and T^b being generators of the $SU(2) \subset G$, we have

$$\text{tr}_{(r)}(T^a T^b) = \delta^{ab} \times \sum_{i=1}^n \frac{1}{3} I(I+1)(2I+1), \quad (\text{S.7})$$

cf. eq. (S.6).

Now, let us suppose that the Lie group G is *simple*, that is, all its generators are related to each other by the symmetry G itself. In this case, for any complete representation (r) ,

$$\text{tr}_{(r)}(T^a T^b) = R(r) \times \delta^{ab} \quad (\text{S.8})$$

with the same index $R(r)$ for any two generators T^a and T^b of G , hence eq. (3), *Q.E.D.*

Problem 1(c):

From the $SU(2) \subset SU(N)$ point of view, the fundamental representation \mathbf{N} of the $SU(N)$ decomposes into a doublet plus $(N - 2)$ singlets,

$$\mathbf{N} = \mathbf{2} + (N - 2) \times \mathbf{1} \equiv (I = \frac{1}{2}) + (N - 2) \times (I = 0), \quad (\text{S.9})$$

hence according to eq. (3),

$$R(\mathbf{N}) = R(I = \frac{1}{2}) + (N - 2) \times R(I = 0) = \frac{1}{2} + (N - 2) \times 0 = \frac{1}{2}$$

and consequently

$$C(\mathbf{N}) = R(\mathbf{N}) \times \frac{\dim(G)}{\dim(\mathbf{N})} = \frac{1}{2} \times \frac{N^2 - 1}{N} = \frac{N^2 - 1}{2N} \quad (4)$$

Now consider the adjoint representation of the $SU(N)$. Let us form a tensor product of the fundamental representation \mathbf{N} and the conjugate (anti-fundamental) representation $\overline{\mathbf{N}}$. Given the transformation laws

$$\begin{aligned} \Psi &\rightarrow U\Psi, & i.e. & \Psi'_j = U_j^k \Psi_k, \\ \overline{\Psi} &\rightarrow \overline{\Psi}U^\dagger, & i.e. & \overline{\Psi}^\ell = \overline{\Psi}^m U_m^{*\ell}, \end{aligned}$$

it follows that the tensor product is an $N \times N$ matrix Φ_j^k which transforms according to

$$\Phi' = U\Phi U^\dagger \quad i.e. \quad \Phi_j'^\ell = U_j^k \Phi_k^m U_m^{*\ell}. \quad (5)$$

Hence, forms a reducible representation of the $SU(N)$, namely the tensor sum of the adjoint representation (the traceless part of Φ) plus the trivial singlet representation (the trace $\text{tr}(\Phi)$). In other words,

$$\mathbf{N} \otimes \overline{\mathbf{N}} = \text{Adj} \oplus \mathbf{1} \quad (\text{S.10}).$$

In $SU(2)$, $\overline{\mathbf{2}} = \mathbf{2}$, and hence from the $SU(2) \subset SU(N)$ point of view, both the fundamental and

the anti-fundamental representations of the $SU(N)$ decompose according to eq. (S.9). Therefore,

$$\begin{aligned}
[\text{Adj} + \mathbf{1}]_{SU(N)} &= [\mathbf{N} \otimes \overline{\mathbf{N}}]_{SU(N)} \\
&= [\mathbf{2} + (N-2) \times \mathbf{1}]_{SU(2)} \otimes [\mathbf{2} + (N-2) \times \mathbf{1}]_{SU(2)} \\
&= [(\mathbf{2} \otimes \mathbf{2}) + 2(N-2) \times (\mathbf{2} \otimes \mathbf{1}) + (N-2)^2 \times (\mathbf{1} \otimes \mathbf{1})]_{SU(2)} \quad (\text{S.11}) \\
&= [\mathbf{3} + \mathbf{1} + 2(N-2) \times \mathbf{2} + (N-2)^2 \times \mathbf{1}]_{SU(2)}, \\
i.e. \quad [\text{Adj}]_{SU(N)} &= [\mathbf{3} + 2(N-2) \times \mathbf{2} + (N-2)^2 \times \mathbf{1}]_{SU(2)},
\end{aligned}$$

and consequently

$$\begin{aligned}
R(\text{Adj}) &= I(\mathbf{3}) + 2(N-2) \times I(\mathbf{2}) + (N-2)^2 \times I(\mathbf{1}) \\
&= 2 + 2(N-2) \times \frac{1}{2} + (N-2)^2 \times 0 = N.
\end{aligned} \quad (\text{S.12})$$

Finally,

$$C(G) \stackrel{\text{def}}{=} C(\text{Adj}(G)) = R(\text{Adj}) \times \frac{\dim(G)}{\dim(G)} = R(\text{Adj}) = N. \quad (\text{S.13})$$

Problem 1(d):

Consider the two-index symmetric tensor $S_{(ij)}$ representation of the $SU(N)$ symmetry group. Denote the index $i = \alpha$ if $i = 1, 2$ or $i = \mu$ if $i = 3, 4, \dots, N$ and likewise $j = \beta$ if $j = 1, 2$ and $j = \nu$ if $j = 3, 4, \dots, N$. Thus, the complete set of independent $S_{(ij)}$ decomposes into $S_{(\alpha\beta)}$, $S_{\alpha,\mu} \equiv S_{\mu,\alpha}$ and $S_{(\mu\nu)}$. The $SU(2) \subset SU(N)$ acts on indices $\alpha, \beta = 1, 2$ and ignores indices $\mu, \nu = 3, 4, \dots, N$. Hence, from the $SU(2)$ point of view, the symmetric tensor of the $SU(N)$ decomposes into one 2-index symmetric tensor $S_{(\alpha\beta)}$, plus $(N-2)$ doublets $S_{\alpha,\mu}$, plus $(N-1)(N-2)/2$ singlets $S_{(\mu\nu)}$. Consequently,

$$\begin{aligned}
R(S) &= R(\mathbf{3}) + (N-2) \times R(\mathbf{2}) + \frac{1}{2}(N-1)(N-2) \times R(\mathbf{1}) \\
&= 2 + (N-2) \times \frac{1}{2} + \frac{1}{2}(N-1)(N-2) \times 0 = \frac{1}{2}(N+2),
\end{aligned} \quad (\text{S.14})$$

and hence

$$C(S) = R(S) \times \frac{\dim(G)}{\dim(S)} = \frac{N+2}{2} \times \frac{N^2-1}{\frac{1}{2}N(N+1)} = \frac{N^2+N-2}{N}. \quad (\text{S.15})$$

Similarly, the two-index anti-symmetric tensor $A_{[ij]}$ decomposes into one anti-symmetric $SU(2)$ tensor $A_{[\alpha\beta]}$, plus $(N-2)$ $SU(2)$ doublets $A_{\alpha,\mu}$, plus $(N-2)(N-3)/2$ $SU(2)$ singlets $A_{[\mu\nu]}$.

The new part of this exercise is summing / averaging over the color indices. By hermiticity of the Lie Algebra matrices T^a , we have

$$\left((T^a)^i_j (T^a)^k_\ell \right)^* = (T^a)^j_i (T^a)^\ell_k = (T^b)^j_i (T^b)^\ell_k \quad (\text{S.20})$$

— note the implicit summation over a or b — and hence

$$\begin{aligned} \sum_{i,j,k,\ell} \left| (T^a)^i_j (T^a)^k_\ell \right|^2 &= \sum_{i,j,k,\ell} (T^a)^i_j (T^a)^k_\ell \times (T^b)^j_i (T^b)^\ell_k \\ &= \sum_{ij} (T^a)^i_j (T^b)^j_i \times \sum_{k,\ell} (T^a)^k_\ell (T^b)^\ell_k \\ &= \text{tr}(T^a T^b) \times \text{tr}(T^a T^b) \end{aligned} \quad (\text{S.21})$$

For the moment, let us consider ‘quarks’ belonging to some generic representation (r) of some generic gauge group G . In such a generic case, $\text{tr}(T^a T^b) = R(r) \times \delta^{ab}$ where $R(r)$ is the index of the representation (r), *cf.* the previous homework, and hence

$$\sum_{a,b} \text{tr}(T^a T^b) \times \text{tr}(T^a T^b) = R^2(r) \times \sum_{a,b} \delta^{ab} \delta^{ab} = R^2(r) \times \dim(G). \quad (\text{S.22})$$

Substituting this formula into eq. (S.21) then gives

$$\sum_{i,j,k,\ell} \left| (T^a)^i_j (T^a)^k_\ell \right|^2 = R^2(r) \times \dim(G),$$

or, for the average over the initial ‘colors’ i and j ,

$$\frac{1}{\dim^2(r)} \sum_{i,j} \sum_{k,\ell} \left| (T^a)^i_j (T^a)^k_\ell \right|^2 = \frac{R^2(r) \dim(G)}{\dim^2(r)} = \frac{C^2(r)}{\dim(G)}. \quad (\text{S.23})$$

Specializing to the ‘quarks’ in the fundamental representation of an $SU(N)$ gauge group, we have $R(r) = \frac{1}{2}$, $\dim(r) = N$ and $\dim(G) = N^2 - 1$, hence eq. (S.23) evaluates to $(N^2 - 1)/(4N^2)$; for the actual quarks, $N = 3$ and the color sum / average (S.23) gives $2/9$.

Altogether, $|\mathcal{M}|^2$ summed / averaged over both spins and colors of all the fermions is

$$\frac{2}{9} \times g^4 (1 + \cos^2 \theta_{\text{c.m.}}) \quad (\text{S.24})$$

and hence the total cross section

$$\sigma(u\bar{u} \rightarrow d\bar{d}) = \frac{8\pi\alpha_{\text{QCD}}^2}{27E_{\text{c.m.}}^2}. \quad (\text{S.25})$$

Problem 3(a):

Classically,

$$\mathcal{L} = \mathcal{L}_{\text{YM}} + D^\mu \Phi^\dagger D_\mu \Phi - V(\Phi^\dagger, \Phi) \quad (\text{S.26})$$

where

$$D_\mu \Phi_i = \partial_\mu \Phi_i + igA_\mu^a (T_{(r)}^a)_i^j \Phi_j, \quad D^\mu \Phi^{*i} = \partial^\mu \Phi^{*i} - igA^{a\mu} \Phi^{*j} (T_{(r)}^a)_j^i, \quad (\text{S.27})$$

and $V(\Phi^\dagger, \Phi)$ is some kind of a G -invariant potential. For Φ_i in the fundamental representation of an $SU(N)$ gauge group, there is only one independent G -invariant function of Φ_i , namely $\Phi^\dagger \Phi \equiv \Phi^{*i} \Phi_i$, hence $V = V(\Phi^\dagger \Phi)$. Furthermore, for renormalizability's sake, V should be a polynomial of degree 4 (or less), thus

$$V(\Phi) = m^2 (\Phi^\dagger \Phi) + \frac{1}{8} \lambda (\Phi^\dagger \Phi)^2. \quad (\text{S.28})$$

For more complicated representations of the gauge group, the scalar potential may include additional, independent G -invariant terms. For example, for $\Phi_{[ij]}$ in the anti-symmetric tensor representation, we generally have

$$V = \frac{1}{2} m^2 \Phi^{*ij} \Phi_{ij} + \frac{1}{32} \lambda (\Phi^{*ij} \Phi_{ij})^2 + \frac{1}{8} \lambda' \Phi^{*ij} \Phi_{jk} \Phi^{*k\ell} \Phi_{\ell i}. \quad (\text{S.29})$$

However, in order to keep this solution as simple as possible, I shall henceforth assume the scalar potential to have form (S.28) regardless of specifics of the representation (r).

In the quantum field theory, the net Lagrangian comprises the classical terms (S.26) plus the ghost Lagrangian, the gauge fixing terms, and the whole slew of counterterms. Generally, all terms pertaining only to the gauge fields and ghost fields have exactly the same form as in the fermionic QCD discussed in class. Consequently, the ‘gluon’ propagator, the three-gluon and four-gluon vertices, the ghost propagator and the ghost vertex are exactly as discussed in class, so I need not repeat them here. The Feynman rules peculiar to the scalar QCD are those pertaining to the scalar fields, thus

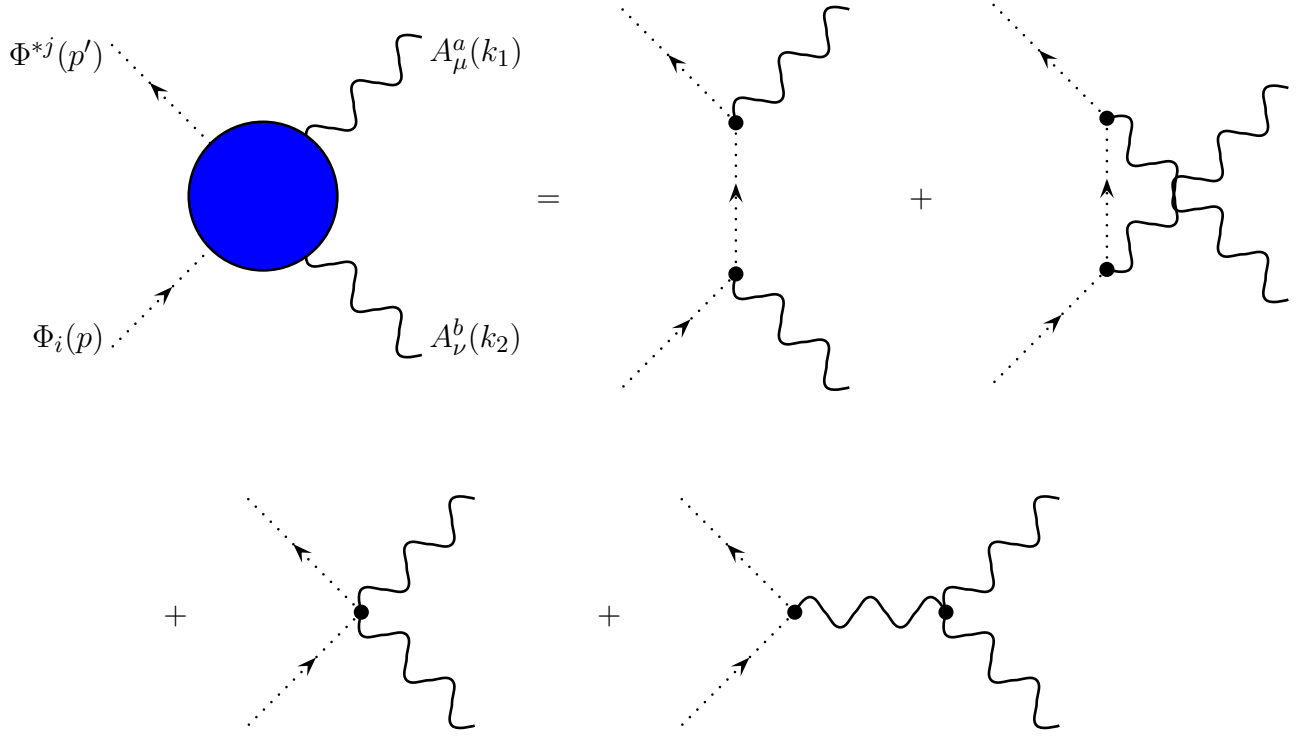
$$\begin{aligned}
 \Phi_i \cdots \cdots \cdots \Phi^{*j} &= \frac{i\delta_i^j}{p^2 - m^2 + i0}, \\
 \begin{array}{c} \Phi_i \\ \searrow \\ \bullet \\ \nearrow \\ \Phi_j \end{array} \begin{array}{c} \Phi^{*k} \\ \nearrow \\ \bullet \\ \searrow \\ \Phi^{*\ell} \end{array} &= -i\lambda(\delta_i^k \delta_j^\ell + \delta_i^\ell \delta_j^k), \\
 \begin{array}{c} \Phi^{*j}(-p') \\ \searrow \\ \bullet \\ \nearrow \\ \Phi_i(p) \end{array} \begin{array}{c} A_\mu^a \end{array} &= ig(p + p')_\mu (T_{(r)}^a)^j_i, \\
 \begin{array}{c} \Phi^{*j} \\ \searrow \\ \bullet \\ \nearrow \\ \Phi_i \end{array} \begin{array}{c} A_\mu^a \\ \nearrow \\ \bullet \\ \searrow \\ A_\nu^b \end{array} &= -ig^2 g_{\mu\nu} \{T_{(r)}^a, T_{(r)}^b\}^j_i.
 \end{aligned}$$

Note the anticommutator of the group generators in the two-scalar two-gluon vertex: It follows from permutations of the two gluon lines.

In addition, there are also the counterterm vertices involving the scalar fields, but they not germane to the present exercise.

Problem 3(b):

There are four tree-level Feynman diagrams for the $\Phi\Phi^* \rightarrow gg$ annihilation process,



(S.30)

so the net tree-level amplitude is $\mathcal{M} = \mathcal{M}^{\mu\nu} \times e_{1\mu}^* e_{2\nu}^*$ where

$$\begin{aligned}
 \mathcal{M}^{\mu\nu} = & \frac{g^2}{(p-k_1)^2 - m^2} (k_2 - 2p')^\nu (2p - k_1)^\mu (T^b T^a)_i^j \\
 & + \frac{g^2}{(p-k_2)^2 - m^2} (k_1 - 2p')^\mu (2p - k_2)^\nu (T^a T^b)_i^j \\
 & - g^2 g^{\mu\nu} \{T^a, T^b\}_i^j \\
 & - \frac{ig^2}{(k_1 + k_2)^2} (p - p')_\lambda (T^c)_i^j \\
 & \quad \times f^{abc} (g^{\mu\nu} (k_1 - k_2)^\lambda + g^{\nu\lambda} (2k_2 + k_1)^\mu + g^{\lambda\mu} (-2k_1 - k_2)^\nu).
 \end{aligned}
 \tag{S.31}$$

Problem 3(c):

Our task is to verify that the amplitude (S.31) satisfies

$$k_1^\mu e_2^\nu \mathcal{M}_{\mu\nu} = 0 \quad (\text{S.32})$$

provided $e_2^\nu k_{2\nu} = 0$ and all external momenta are on shell. Let $\mathcal{M}_{1,2,3,4}^{\mu\nu}$ denote respectively the four terms on the right hand side of eq. (S.31). Then it is easy to show that for $p^2 = p'^2 = m^2$

$$\begin{aligned} k_{1\mu} \mathcal{M}_1^{\mu\nu} &= g^2 (2p' - k_2)^\nu (T^b T^a)^j_i, \\ k_{1\mu} \mathcal{M}_2^{\mu\nu} &= g^2 (2p - k_2)^\nu (T^a T^b)^j_i, \\ k_{1\mu} \mathcal{M}_3^{\mu\nu} &= -g^2 k_1^\nu \{T^a, T^b\}^j_i, \end{aligned} \quad (\text{S.33})$$

and therefore

$$\begin{aligned} k_{1\mu} (\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)^{\mu\nu} &= g^2 (2p' - k_2 - k_1)^\nu (T^b T^a)^j_i + g^2 (2p - k_1 - k_2)^\nu (T^a T^b)^j_i \\ &= g^2 (p - p')^\nu \times (-T^b T^a + T^a T^b)^j_i \\ &= g^2 (p - p')^\nu \times i f^{abc} (T^c)^j_i. \end{aligned} \quad (\text{S.34})$$

At the same time, for $k_1^2 = k_2^2 = 0$ we have

$$\begin{aligned} k_{1\mu} \times &\left(g^{\mu\nu} (k_1 - k_2)^\lambda + g^{\nu\lambda} (2k_2 + k_1)^\mu + g^{\lambda\mu} (-2k_1 - k_2)^\nu \right) \\ &= (k_1 + k_2)^2 g^{\nu\lambda} + k_2^\nu k_2^\lambda - (k_1 + k_2)^\nu (k_1 + k_2)^\lambda \end{aligned} \quad (\text{S.35})$$

and hence

$$k_{1\mu} \mathcal{M}_4^{\mu\nu} = i g^2 (p - p')_\lambda \left[g^{\lambda\nu} + \frac{k_2^\lambda k_2^\nu}{(k_1 + k_2)^2} + 0 \right] \times i f^{abc} (T^c)^j_i. \quad (\text{S.36})$$

because on shell $(k_1 + k_2)^\lambda \times (p - p')_\lambda = (p + p')^\lambda (p - p')_\lambda = p^2 - p'^2 = 0$.

Consequently, in light of eq. (S.34), most terms in $k_{1\mu} \mathcal{M}^{\mu\nu}$ cancel out, except for the second term in eq. (S.36), thus

$$k_{1\mu} \mathcal{M}^{\mu\nu} = g^2 \frac{k_2 (p - p')}{(k_1 + k_2)^2} \times i f^{abc} (T^c)^j_i \times k_2^\nu. \quad (\text{S.37})$$

Similarly to the fermionic QCD discussed in class, this amplitude does not quite vanish but is proportional to the k_2^ν . Consequently, $k_{1\mu} \mathcal{M}^{\mu\nu} e_{2\nu}^* = 0$ when the second gluon is transversely

polarized, $k_2 e_2 = 0$, but not if the other gluon's polarization is longitudinal. And this is in accordance to the weak form of Ward Identity: *On-shell amplitudes involving one longitudinal gluon vanish, but only if all the other gluons are transverse.* $\mathcal{Q.E.D.}$

Problem 4(a):

Consider the amplitude (S.31): For transverse gauge bosons $(e_1^* k_1) = (e_2^* k_2) = 0$, we have

$$\begin{aligned}
\mathcal{M} \equiv \mathcal{M}^{\mu\nu} e_{1\mu}^* e_{2\nu}^* &= - \frac{4g^2}{t - m^2} (e_1^* p)(e_2^* p') \times (T^b T^a)^j_i \\
&- \frac{4g^2}{u - m^2} (e_1^* p')(e_2^* p) \times (T^a T^b)^j_i \\
&- g^2 (e_1^* e_2^*) \times \{T^a, T^b\}^j_i \\
&- \frac{ig^2}{s} [(u - t)(e_1^* e_2^*) + 2(e_1^* k_2)(e_2^*(p - p')) - 2(e_2^* k_1)(e_1^*(p - p'))] \\
&\quad \times f^{abc} (T^c)^j_i
\end{aligned} \tag{S.38}$$

where s , t and u are Mandelstamm's kinematic variables and

$$(u - t) = (p - p')_\lambda (k_1 - k_2)^\lambda.$$

Clearly,

$$\begin{aligned}
(T^a T^b) &= \frac{1}{2} \{T^a, T^b\} + \frac{1}{2} [T^a, T^b], \\
(T^b T^a) &= \frac{1}{2} \{T^a, T^b\} - \frac{1}{2} [T^a, T^b], \\
if^{abc} T^c &= [T^a, T^b],
\end{aligned} \tag{S.39}$$

so indeed, every term in eq. (S.31) can be written in the form (6). Specifically,

$$\begin{aligned}
F &= - \frac{2g^2 (e_1^* p)(e_2^* p')}{t - m^2} - \frac{2g^2 (e_1^* p')(e_2^* p)}{u - m^2} - g^2 (e_1^* e_2^*), \\
iG &= + \frac{2g^2 (e_1^* p)(e_2^* p')}{t - m^2} - \frac{2g^2 (e_1^* p')(e_2^* p)}{u - m^2} \\
&\quad - \frac{g^2}{s} [(u - t)(e_1^* e_2^*) + 2(e_1^* k_2)(e_2^*(p - p')) - 2(e_2^* k_1)(e_1^*(p - p'))].
\end{aligned} \tag{S.40}$$

Problem 4(b):

First, let us average over the scalar particles' color indices $i, j = 1, 2, \dots, \dim(r)$. For fixed gauge bosons a and b , let

$$M = F\{T_{(r)}^a, T_{(r)}^b\} + iG [T_{(r)}^a, T_{(r)}^b] \quad (\text{S.41})$$

be a matrix (in the representation (r) of the gauge group) whose elements M_i^j are annihilation amplitudes (6) for the scalar particles Φ_i and Φ^{*j} of specific colors i, j . Then averaging over those colors gives

$$\frac{1}{\dim^2(r)} \sum_{i,j} |M_i^j|^2 = \frac{1}{\dim^2(r)} \sum_{i,j} M_i^j (M^\dagger)^i_j = \frac{1}{\dim^2(r)} \text{tr}(MM^\dagger). \quad (\text{S.42})$$

For the specific form (S.41) of the matrix M , we write

$$\begin{aligned} M &= (F + iG) T_{(r)}^a T_{(r)}^b + (F - iG) T_{(r)}^b T_{(r)}^a, \\ M^\dagger &= (F + iG)^* T_{(r)}^b T_{(r)}^a + (F - iG)^* T_{(r)}^a T_{(r)}^b, \end{aligned} \quad (\text{S.43})$$

and therefore

$$\begin{aligned} \text{tr}(MM^\dagger) &= |F + iG|^2 \text{tr}_{(r)}(T^a T^b T^b T^a) + (F - iG)(F + iG)^* \text{tr}_{(r)}(T^b T^a T^b T^a) \\ &\quad + (F + iG)(F - iG)^* \text{tr}_{(r)}(T^a T^b T^a T^b) + |F - iG|^2 \text{tr}_{(r)}(T^b T^a T^a T^b) \\ &= 2(|F|^2 + |G|^2) \text{tr}_{(r)}(T^a T^a T^b T^b) + 2(|F|^2 - |G|^2) \text{tr}_{(r)}(T^a T^b T^a T^b) \\ &= 4|F|^2 \text{tr}_{(r)}(T^a T^a T^b T^b) + 2(|G|^2 - |F|^2) \text{tr}_{(r)}(T^a [T^a, T^b] T^b). \end{aligned} \quad (\text{S.44})$$

where the second equality follows from the cyclic symmetry of the traces.

Our next step is to sum over the color indices a and b of the gauge bosons. In the context of eq. (S.44), we have

$$\sum_{a,b} \text{tr}_{(r)}(T^a T^a T^b T^b) = \text{tr}_{(r)} \left(\left(\sum_a T^a T^a \right) \left(\sum_b T^b T^b \right) \right) = \text{tr}_{(r)} \left(\hat{C}_2 \hat{C}_2 \right) = C^2(r) \times \dim(r) \quad (\text{S.45})$$

and

$$\begin{aligned}
\sum_{a,b} \text{tr}_{(r)}(T^a[T^a, T^b]T^b) &= \sum_{a,b} \sum_c i f^{abc} \text{tr}_{(r)}(T^a T^c T^b) \\
&= \frac{1}{2} \sum_{a,b,c} i f^{abc} \text{tr}_{(r)}(T^a T^c T^b - T^a T^b T^c) \\
&= \frac{1}{2} \sum_{a,b,c} i f^{abc} \sum_d i f^{cbd} \text{tr}_{(r)}(T^a T^d) \\
&= \frac{1}{2} \sum_{a,b,c,d} (i f^{abc})(i f^{dbc}) \times R(r) \delta^{ad} \\
&= \frac{1}{2} R(r) \sum_a \sum_{b,c} \left(i f^{abc} = (T_{(\text{adj})}^a)^{bc} \right) \left(i f^{acb} = (T_{(\text{adj})}^a)^{cb} \right) \\
&= \frac{1}{2} R(r) \sum_a \text{tr} \left(T_{(\text{adj})}^a T_{(\text{adj})}^a \right) \\
&= \frac{1}{2} R(r) \times C(G) \dim(G) \\
&= \frac{1}{2} C(G) C(r) \dim(r).
\end{aligned} \tag{S.46}$$

Therefore,

$$\sum_{a,b} \text{tr}(MM^\dagger) = C(r) \dim(r) \times [4C(r)|F|^2 + C(G)(|G|^2 - |F|^2)] \tag{S.47}$$

and hence in light of eq. (S.42),

$$\frac{1}{\dim^2(r)} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{C(r)}{\dim(r)} \times (4C(r)|F|^2 + C(\text{adj})(|G|^2 - |F|^2)). \tag{7}$$

Eq. (8) follows from this as a special case. $\mathcal{Q.E.D.}$

Problem 4(c):

Let us take a closer look at eqs. (S.40). In the center of mass frame, $\mathbf{p}' = -\mathbf{p}$, $\mathbf{k}_2 = -\mathbf{k}_1$, and the vector bosons' polarizations $e_{1,2}^\mu$ are purely spatial and transverse, $e_{1,2}^0 = 0$ and $\mathbf{k}_{1,2} \cdot \mathbf{e}_{1,2} = 0$.

Consequently, eqs. (S.40) simplify to

$$\begin{aligned}
F &= 2g^2(\mathbf{e}_1^* \mathbf{p})(\mathbf{e}_2^* \mathbf{p}) \left(\frac{1}{t-m^2} + \frac{1}{u-m^2} \right) + g^2(\mathbf{e}_1^* \mathbf{e}_2^*), \\
G &= 2ig^2(\mathbf{e}_1^* \mathbf{p})(\mathbf{e}_2^* \mathbf{p}) \left(\frac{1}{t-m^2} - \frac{1}{u-m^2} \right) - ig^2 \frac{u-t}{s} (\mathbf{e}_1^* \mathbf{e}_2^*).
\end{aligned} \tag{S.48}$$

Furthermore, in the center of mass frame $E = E' = \omega_1 = \omega_2$, $|\mathbf{k}| = \omega = E$, $|\mathbf{p}| = \beta E$,

$$s = 4E^2, \quad t - m^2 = -2E^2(1 - \beta \cos \theta), \quad u - m^2 = -2E^2(1 + \beta \cos \theta),$$

hence

$$\begin{aligned} \frac{u-t}{s} &= \beta \cos \theta, \\ \frac{1}{t-m^2} + \frac{1}{u-m^2} &= \frac{-1}{m^2 + \mathbf{p}^2 \sin^2 \theta}, \\ \frac{1}{t-m^2} - \frac{1}{u-m^2} &= \frac{-\beta \cos \theta}{m^2 + \mathbf{p}^2 \sin^2 \theta}, \end{aligned}$$

and therefore

$$\begin{aligned} F &= g^2 \left((\mathbf{e}_1^* \mathbf{e}_2^*) - \frac{2(\mathbf{e}_1^* \mathbf{p})(\mathbf{e}_2^* \mathbf{p})}{m^2 + \mathbf{p}^2 \sin^2 \theta} \right), \\ G &= -ig^2 \left((\mathbf{e}_1^* \mathbf{e}_2^*) + \frac{2(\mathbf{e}_1^* \mathbf{p})(\mathbf{e}_2^* \mathbf{p})}{m^2 + \mathbf{p}^2 \sin^2 \theta} \right) \times \beta \cos \theta. \end{aligned} \tag{S.49}$$

Now consider the gluons' polarization vectors. For the problem at hand it is easier to use linear polarizations for which the \mathbf{e}_1 and \mathbf{e}_2 are real unit vectors. Specifically, for each gluon there is a choice of two transverse \mathbf{e} one parallel to the (\mathbf{p}, \mathbf{k}) plane and one perpendicular to it. In the coordinate system where

$$\mathbf{p} = \beta E(0, 0, 1) \quad \text{and} \quad \mathbf{k} = E(\sin \theta, 0, \cos \theta), \tag{S.50}$$

the two polarization vectors are

$$\mathbf{e}_{\parallel} = (-\cos \theta, 0, +\sin \theta) \quad \text{and} \quad \mathbf{e}_{\perp} = (0, 1, 0). \tag{S.51}$$

For these vectors

$$(\mathbf{p} \mathbf{e}_{\parallel}) = \beta E \sin \theta, \quad (\mathbf{p} \mathbf{e}_{\perp}) = 0. \tag{S.52}$$

and hence according to eqs. (S.49),

$$\text{for } \mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_{\perp}, \quad F = g^2 \quad \text{and} \quad G = -ig^2 \beta \cos \theta, \tag{S.53}$$

$$\text{for } \mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_{\parallel}, \quad F = g^2(1 - 2A) \quad \text{and} \quad G = -ig^2(1 + 2A)\beta \cos \theta \quad (\text{S.54})$$

where

$$A = \frac{\mathbf{p}^2 \sin^2 \theta}{m^2 + \mathbf{p}^2 \sin^2 \theta}, \quad (\text{S.55})$$

and finally

$$\text{for } \mathbf{e}_1 = \mathbf{e}_{\perp}, \mathbf{e}_2 = \mathbf{e}_{\parallel} \text{ or } \textit{vice versa}, \quad F = G = 0. \quad (\text{S.56})$$

Problem 4(d):

According to eq. (S.56), the two gauge bosons produced in the $\Phi\Phi^*$ annihilation must have similar polarizations: either both are polarized \parallel to the (\mathbf{p}, \mathbf{k}) plane of scattering or both are polarized \perp to the plane. Consequently, there are only two polarized partial cross sections to consider, namely

$$\begin{aligned} \left(\frac{d\sigma(\perp)}{d\Omega} \right)_{\text{c.m.}} &= \frac{g^4}{64\pi^2 E_{\text{c.m.}}^2 \beta} \frac{C(r)}{\text{dim}(r)} \left(4C(r) - (1 - \beta^2 \cos^2 \theta)C(G) \right), \\ \left(\frac{d\sigma(\parallel)}{d\Omega} \right)_{\text{c.m.}} &= \frac{g^4}{64\pi^2 E_{\text{c.m.}}^2 \beta} \frac{C(r)}{\text{dim}(r)} \left(4C(r)(1 - 2A)^2 - C(G)((1 - 2A)^2 - \beta^2(1 + 2A)^2 \cos^2 \theta) \right). \end{aligned} \quad (\text{S.57})$$

Note that the angular dependence of the \parallel polarized partial cross section is more complicated than it looks because A is θ -dependent according to eq. (S.55).

In the limit of non-relativistic scalar particles, $\beta \ll 1$ leads to $A \ll 1$ and hence to the expected isotropy and polarization independence of the annihilation cross-section,

$$\left(\frac{d\sigma(\parallel)}{d\Omega} \right)_{\text{c.m.}} \approx \left(\frac{d\sigma(\perp)}{d\Omega} \right)_{\text{c.m.}} \approx \frac{g^4}{256\pi^2 m^2 \beta} \frac{C(r)(4C(r) - C(G))}{\text{dim}(r)}. \quad (\text{S.58})$$

In the opposite limit of ultra-relativistic scalars, $\beta \approx 1$ leads to $A \approx 1$ (except for $\theta \approx 0$) and therefore

$$\begin{aligned} \left(\frac{d\sigma(\perp)}{d\Omega} \right)_{\text{c.m.}} &\approx \frac{g^4}{16\pi^2 E_{\text{c.m.}}^2} \frac{C^2(r)}{\text{dim}(r)}, \\ \left(\frac{d\sigma(\parallel)}{d\Omega} \right)_{\text{c.m.}} &\approx \frac{g^4}{16\pi^2 E_{\text{c.m.}}^2} \frac{C(r)}{\text{dim}(r)} \left[C(r) + C(G) \frac{9 \cos^2 \theta - 1}{4} \right]. \end{aligned} \quad (\text{S.59})$$