Problem 3(a):
In this problem, scalars belonging to two distinct multiplets of the $SU(2) \times U(1)$ symmetry develop non-zero vacuum expectation values (VEVs). We can use either multiplet to fix a unitary gauge, so let us use the Standard Higgs doublet $H$ and demand that

$$
\begin{pmatrix}
H_1(x) \\
H_2(x)
\end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \text{real } h(x) \end{pmatrix} \Rightarrow \langle H \rangle = \frac{v_h}{\sqrt{2}} \begin{pmatrix} 0 \\
1 \end{pmatrix}. \quad (S.1)
$$

This VEV breaks the $T^1, T^2$ and $T^3 - Y$ generators of the $SU(2) \times U(1)$ gauge symmetry, but the remaining generator $Q = T^3 + Y$ remains unbroken. In the unitary gauge defined by eq. (S.1), the real $\varphi^a(x)$ fields become

$$
\Phi(x) = \begin{pmatrix} \varphi^+(x) \\
\varphi^0(x) \\
\varphi^-(x) \end{pmatrix}, \quad (\varphi^+)^* = \varphi^- \quad (S.2)
$$

where the superscript of each component field is its electric charge $Q$.

The problem does not specify the scalar potential $V(H, \varphi)$ but we are told that the VEVs $\langle H \rangle$ and $\langle \varphi \rangle$ are 'aligned' such that the photon remains massless. This means that only fields neutral with respect to $Q$ develop VEVs, hence in terms of eq. (S.2), $\langle \varphi^+ \rangle = \langle \varphi^- \rangle = 0$ but generally $\langle \varphi^0 \rangle = v_\varphi \neq 0$. In other words,

$$
\langle \Phi \rangle = v_\varphi \begin{pmatrix} 0 \\
1 \\
0 \end{pmatrix}. \quad (S.3)
$$

Now consider the covariant derivatives of the scalar VEVs,

$$
D_\mu \langle H \rangle = \frac{v_h}{\sqrt{2}} \begin{pmatrix} i g \left( W^1_\mu + i W^2_\mu \right) \\
- i g \left( W^3_\mu + i g' B_\mu \right) \end{pmatrix},
$$

$$
D_\mu \langle \varphi \rangle = v_\varphi \begin{pmatrix} g \left( W^1_\mu + i W^2_\mu \right) \\
0 \\
0 \end{pmatrix}. \quad (S.4)
$$
The mass terms for the vector fields come from
\[ \mathcal{L}^{\text{mass}} = D_{\mu} \langle H \rangle^\dagger D^\mu \langle H \rangle + \frac{1}{2} D_{\mu} \langle \Phi \rangle^\dagger D^\mu \langle \Phi \rangle \]
\[ = \frac{v^2}{2} \left[ \frac{1}{4} g^2 |W^1_{\mu}|^2 + \frac{1}{4} (g W^3_{\mu} - g' B_{\mu})^2 \right] + \frac{v'^2}{2} \times g^2 |W^1_{\mu} + iW^2_{\mu}|^2. \] (S.5)

Diagonalizing the resulting mass matrix, we obtain the eigens\( \text{fields} \)
\[ W_{\mu}^\pm = \frac{W^1_{\mu} \pm iW^2_{\mu}}{\sqrt{2}}, \]
\[ Z_{\mu} = \frac{g W^3_{\mu} - g' B_{\mu}}{\sqrt{g^2 + g'^2}}, \]
\[ \equiv \cos \theta \times W^3_{\mu} - \sin \theta \times B_{\mu}, \]
\[ A_{\mu} = \sin \theta \times W^3_{\mu} + \cos \theta \times B_{\mu}, \] (S.6)

where
\[ \theta = \arctan \frac{g'}{g} \] (S.7)
is the weak mixing angle, and the eigenmasses
\[ M^2_{W} = \frac{v^2 g^2}{4} + \frac{v'^2 g^2}{4}, \]
\[ M^2_{Z} = \frac{v^2 (g^2 + g'^2)}{4} + 0, \]
\[ M^2_{A} = 0, \] (S.8)

Note that in eq. (S.5), the triplet VEV \( v_{\phi} \) couples to the charged vector fields \( W^1_{\mu}, W^2_{\mu} \) but not to the neutral fields \( W^3_{\mu}, B_{\mu} \to Z_{\mu}, A_{\mu} \). Consequently, the neutral fields are exactly as in the \( \Phi \)-less Standard Model — same mixing angle, same \( M_Z \), same \( M_A = 0, \) — but the charged fields have a larger mass. Thus,
\[ \frac{M^2_{W}}{M^2_{Z}} > \frac{g^2}{g^2 + g'^2} \equiv \cos^2 \theta. \] (S.9)

Finally, the couplings of the vector fields to the quarks and leptons are completely determined by the \( SU(2) \times U(1) \) group theory. In terms of the original \( W^1_{\mu}, W^2_{\mu} \) and \( B_{\mu} \)
fields,
\[ \mathcal{L}^\Psi = i \overline{\Psi} D_\mu \Psi - g W_\mu^a J_L^{a,\mu} - g' B_\mu J_Y^\mu \]  
(S.10)

where
\[ J_L^{a,\mu} = \overline{\Psi} \frac{1 - \gamma^5}{2} \gamma^\mu \gamma^a \Psi \]  
(S.11)
is the left-handed isospin current, and
\[ J_Y^\mu = \overline{\Psi} Y^\mu \Psi \]  
(S.12)
is the hypercharge current, which involves fermions of both left and right chiralities according
to \( Y = \frac{1 - \gamma^5}{2} Y_L + \frac{1 + \gamma^5}{2} Y_R \). Hence, in terms of the eigenfields (S.6) of the mass matrix,
\[ \mathcal{L}^\Psi \supset g W_\mu^a J_L^{a,\mu} - g' B_\mu J_Y^\mu = -\frac{g}{\sqrt{2}} \left( W_\mu^+ J_L^{\mu} + W_\mu^- J_L^{\mu} \right) - g' Z_\mu J_Z^\mu - e A_\mu J_A^\mu \]  
(S.13)

where
\[ J_L^{++,\mu} = J_L^{1,\mu} \pm i J_L^{2,\mu}, \]
\[ g' J_Z^\mu = \cos \theta \times g J_L^{3,\mu} - \sin \theta \times g' J_Y^\mu, \]
\[ e J_A^\mu = \sin \theta \times g J_L^{3,\mu} + \cos \theta \times g' J_Y^\mu. \]  
(S.14)

In light of eq. (S.7), the neutral currents here can be written in a simpler form as
\[ J_A^\mu = J_L^{3,\mu} + J_Y^\mu, \quad J_Z^\mu = J_L^{3,\mu} - \sin^2 \theta \times J_A^\mu, \]  
(S.15)

provided we identify the couplings according to
\[ e = \frac{gg'}{\sqrt{g^2 + g'^2}} = g \sin \theta, \quad \tilde{g} = \sqrt{g^2 + g'^2} = \frac{g}{\cos \theta}. \]  
(S.16)

Note that the currents here depends only on the \( SU(2) \times U(1) \) quantum numbers of quarks
and leptons, and on the eigenfields (S.6) of the vector mass matrix, but they do not depend
on the eigenmasses. The triplet VEV \( \langle \Phi \rangle \) affects the \( W \) mass, but all the eigenfields remain
exactly as in the Standard Model, and therefore their couplings to the quarks and leptons
are exactly Standard.
Problem 3(b):
Physically, the massless vector field \( A_\mu \) is the EM field, and the corresponding current \( J_\mu^A \equiv J_{\text{EM}}^\mu \) is the electric current of quarks and leptons. The rest of the fields \( W^\pm_\mu \), \( Z_\mu \) and currents \( J^\pm_L \), \( J^\mu_Z \) give rise to the weak interactions. Thus,

\[
\mathcal{L}^{\text{weak}} = -\frac{1}{2}W^{\pm}_{\mu \nu}W^{-,-\mu \nu} + M^2_WW^{+,-\mu \nu} - \frac{1}{4}Z_{\mu \nu}Z^{\mu \nu} + \frac{1}{2}M^2_ZZ_\mu Z^\mu \\
- \frac{g}{\sqrt{2}} \left( W^+_\mu J^-_{L,\mu} + W^-_\mu J^+_{L,\mu} \right) - \tilde{g}Z_\mu J^\mu_Z.
\]

(S.17)

Now, let us focus on the low-energy amplitudes without incoming or outgoing vector particles. The internal lines of the Feynman diagrams do include the vector propagators, but in the low-energy regime we may approximate such propagators as

\[
\frac{-ig^{\mu \nu}}{q^2 - M^2} \approx \frac{+ig^{\mu \nu}}{M^2}.
\]

(S.18)

This approximation often fails in loop graphs, but it works well at the tree level when all the momenta are small compared to \( M_W \) or \( M_Z \).

In Lagrangian terms, the approximation (S.18) corresponds to neglecting the kinetic terms for the vector fields compared to the mass terms,

\[
\mathcal{L}^{\text{weak}} \approx M^2_WW^{+,-\mu \nu} + \frac{1}{2}M^2_ZZ_\mu Z^\mu - \frac{g}{\sqrt{2}} \left( W^+_\mu J^-_{L,\mu} + W^-_\mu J^+_{L,\mu} \right) - \tilde{g}Z_\mu J^\mu_Z.
\]

(S.19)

In this formula, the vector fields appear as auxiliary fields with algebraic (\( i.e. \), derivative-less) equations of motions. Consequently, we may solve those equations of motions and plug them back into eq. (S.19); the result is an effective current-current Lagrangian

\[
\mathcal{L}^{\text{eff}} = -\frac{g^2}{2M^2_W} J^{+,-\mu}_L J^-_{L,\mu} - \frac{\tilde{g}^2}{M^2_Z} J^\mu_Z J^\mu_Z.
\]

(S.20)

governing the low-energy weak interactions of quarks and leptons. For historic reasons, this
effective Lagrangian is usually written as
\[ \mathcal{L}^{\text{eff}} = -\frac{4G_F}{\sqrt{2}} \left[ J^+_{L,\mu} J^-_{L,\mu} + \rho J^\mu_{Z} J^\mu_{Z,\mu} \right] \] (1)

where
\[ G_F = \frac{\sqrt{2}g^2}{8M_W^2} \] (S.21)
is the original Fermi’s weak coupling constant, and
\[ \rho = \frac{\tilde{g}^2}{M_Z^2} \bigg/ \frac{g^2}{M_W^2} = \frac{M_W^2}{M_Z^2} \times \frac{1}{\cos^2 \theta} \] (S.22)
is the ratio of neutral-current to charged-current weak interactions.

In the standard model,
\[ G_F = \frac{\sqrt{2}}{2v_h^2} \] (S.23)
and \( \rho = 1 \) because \( M_W = M_Z \times \cos \theta \). Adding the triplet VEV increases the \( W \) mass but does not affect the \( Z \) mass and the mixing angle \( \theta \). Consequently, the Fermi coupling becomes
\[ G_F = \frac{\sqrt{2}}{2v_h^2 + 8v_e^2} \] (S.24)
and the \( \rho \) parameter becomes
\[ \rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta} = \frac{v_h^2 + 4v_e^2}{v_h^2} > 1. \] (S.25)