Problem **3**(a):

In this problem, scalars belonging to two distinct multiplets of the $SU(2)\times U(1)$ symmetry develop non-zero vacuum expectation values (VEVs). We can use either multiplet to fix a unitary gauge, so let us use the Standard Higgs doublet H and demand that

$$\begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \text{real } h(x) \end{pmatrix} \implies \langle H \rangle = \frac{v_h}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{S.1}$$

This VEV breaks the T^1, T^2 and $T^3 - Y$ generators of the $SU(2)\times U(1)$ gauge symmetry, but the remaining generator $Q = T^3 + Y$ remains unbroken. In the unitary gauge defined by eq. (S.1), the real $\varphi^a(x)$ fields become

$$\Phi(x) = \begin{pmatrix} \varphi^+(x) \\ \varphi^0(x) \\ \varphi^-(x) \end{pmatrix}, \qquad (\varphi^+)^* = \varphi^-, \tag{S.2}$$

where the superscript of each component field is its electric charge Q.

The problem does not specify the scalar potential $V(H,\varphi)$ but we are told that the VEVs $\langle H \rangle$ and $\langle \varphi \rangle$ are 'aligned' such that the photon remains massless. This means that only fields neutral with respect to Q develop VEVs, hence in terms of eq. (S.2), $\langle \varphi^+ \rangle = \langle \varphi^- \rangle = 0$ but generally $\langle \varphi^0 \rangle = v_\varphi \neq 0$. In other words,

$$\langle \Phi \rangle = v_\varphi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{S.3}$$

Now consider the covariant derivatives of the scalar VEVs,

$$\begin{aligned} D_\mu \langle H \rangle &= \frac{v_h}{\sqrt{2}} \begin{pmatrix} \frac{ig}{2}(W^1_\mu + iW^2_\mu) \\ -\frac{ig}{2}W^3_\mu + \frac{ig'}{2}B_\mu \end{pmatrix}, \\ D_\mu \langle \varphi \rangle &= v_\varphi \begin{pmatrix} g(W^1_\mu + iW^2_\mu) \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{S.4}$$

The mass terms for the vector fields come from

$$\begin{aligned}
\mathcal{L}^{\text{mass}} &= D_\mu \langle H\rangle^\dagger\, D^\mu \langle H\rangle \;+\; \tfrac{1}{2} D_\mu \langle \Phi\rangle^\dagger\, D^\mu \langle\Phi\rangle \\
&= \frac{v_h^2}{2}\left[\tfrac{1}{4} g^2 |W^1_\mu + iW^2_\mu|^2 \;+ \tfrac{1}{4}(gW^3_\mu - g' B_\mu)^2\right] \;+\; \frac{v_\varphi^2}{2}\times g^2 |W^1_\mu + iW^2_\mu|^2 .
\end{aligned} \tag{S.5}$$

Diagonalizing the resulting mass matrix, we obtain the eigensfields

$$\begin{aligned}
W^\pm_\mu &= \frac{W^1_\mu \pm iW^2_\mu}{\sqrt{2}}\,, \\
Z_\mu &= \frac{gW^3_\mu - g' B_\mu}{\sqrt{g^2 + g'^2}} \\
&\equiv \cos\theta \times W^3_\mu \;-\; \sin\theta \times B_\mu\,, \\
A_\mu &= \sin\theta \times W^3_\mu \;+\; \cos\theta \times B_\mu\,,
\end{aligned} \tag{S.6}$$

where

$$\theta \;=\; \arctan\frac{g'}{g} \tag{S.7}$$

is the weak mixing angle, and the eigenmasses

$$\begin{aligned}
M_W^2 &= \frac{v_h^2 g^2}{4} \;+\; v_\varphi^2 g^2, \\
M_Z^2 &= \frac{v_h^2 (g^2 + g'^2)}{4} \;+\; 0, \\
M_A^2 &= 0.
\end{aligned} \tag{S.8}$$

Note that in eq. (S.5), the triplet VEV v_φ couples to the charged vector fields $W^{1,2}_\mu \to W^\pm_\mu$ but not to the neutral fields $W^3_\mu, B_\mu \to Z_\mu, A_\mu$. Consequently, the neutral fields are exactly as in the Φ–less Standard Model — same mixing angle, same M_Z, same $M_A = 0$, — but the charged fields have a larger mass. Thus,

$$\frac{M_W^2}{M_Z^2} \;>\; \frac{g^2}{g^2 + g'^2} \;\equiv\; \cos^2\theta. \tag{S.9}$$

Finally, the couplings of the vector fields to the quarks and leptons are completely determined by the $SU(2)\times U(1)$ group theory. In terms of the original $W^{1,2,3}_\mu$ and B_μ

fields,

$$\mathcal{L}^{\Psi} \;=\; i\overline{\Psi}\not{D}\Psi \;\supset\; -gW^{a}_{\mu}J^{a,\mu}_{L} \;-\; g'B_{\mu}J^{\mu}_{Y} \tag{S.10}$$

where

$$J^{a,\mu}_{L} \;=\; \overline{\Psi}\frac{1-\gamma^5}{2}\gamma^{\mu}\frac{\tau^a}{2}\Psi \tag{S.11}$$

is the left-handed isospin current, and

$$J^{\mu}_{Y} \;=\; \overline{\Psi}Y\gamma^{\mu}\Psi \tag{S.12}$$

is the hypercharge current, which involves fermions of both left and right chiralities according to $Y = \frac{1-\gamma^5}{2}Y_L + \frac{1+\gamma^2}{2}Y_R$. Hence, in terms of the eigenfields (S.6) of the mass matrix,

$$\mathcal{L}^{\Psi} \supset gW^{a}_{\mu}J^{a,\mu}_{L} \;-\; g'B_{\mu}J^{\mu}_{Y} \;=\; -\frac{g}{\sqrt{2}}\left(W^{+}_{\mu}J^{-,\mu}_{L} + W^{-}_{\mu}J^{+,\mu}_{L}\right) \;-\; \tilde{g}Z_{\mu}J^{\mu}_{Z} \;-\; eA_{\mu}J^{\mu}_{A} \tag{S.13}$$

where

$$\begin{aligned} J^{\pm,\mu}_{L} &= J^{1,\mu}_{L} \;\pm\; iJ^{2,\mu}_{L}, \\ \tilde{g}J^{\mu}_{Z} &= \cos\theta\times gJ^{3,\mu}_{L} \;-\; \sin\theta\times g'J^{\mu}_{Y}, \\ eJ^{\mu}_{A} &= \sin\theta\times gJ^{3,\mu}_{L} \;+\; \cos\theta\times g'J^{\mu}_{Y}. \end{aligned} \tag{S.14}$$

In light of eq. (S.7), the neutral currents here can be written in a simpler form as

$$J^{\mu}_{A} \;=\; J^{3\,mu}_{L} \;+\; J^{\mu}_{Y}, \qquad J^{\mu}_{Z} \;=\; J^{3,\mu}_{L} \;-\; \sin^2\theta\times J^{\mu}_{A}, \tag{S.15}$$

provided we identify the couplings according to

$$e \;=\; \frac{gg'}{\sqrt{g^2+g'^2}} \;=\; g\sin\theta, \qquad \tilde{g} \;=\; \sqrt{g^2+g'^2} \;=\; \frac{g}{\cos\theta}\,. \tag{S.16}$$

Note that the currents here depends only on the $SU(2)\times U(1)$ quantum numbers of quarks and leptons, and on the eigenfields (S.6) of the vector mass matrix, but they do not depend on the eigenmasses. The triplet VEV $\langle\Phi\rangle$ affects the W mass, but all the eigenfields remain exactly as in the Standard Model, and therefore their couplings to the quarks and leptons are exactly Standard.

Problem **3**(b):

Physically, the massless vector field A_μ is the EM field, and the corresponding current $J^\mu_A \equiv J^\mu_{\rm EM}$ is the electric current of quarks and leptons. The rest of the fields $W^\pm_\mu$, Z_μ and currents $J^{\pm,\mu}_L$, J^μ_Z give rise to the weak interactions. Thus,

$$\begin{aligned}\mathcal{L}^{\rm weak} \;=\; & -\tfrac{1}{2}W^+_{\mu\nu}W^{-,\mu\nu} \;+\; M_W^2 W^+_\mu W^{-,\mu} \;-\; \tfrac{1}{4}Z_{\mu\nu}Z^{\mu\nu} \;+\; \tfrac{1}{2}M_Z^2 Z_\mu Z^\mu \\ & -\; \frac{g}{\sqrt{2}}\left(W^+_\mu J^{-,\mu}_L \;+\; W^-_\mu J^{+,\mu}_L\right) \;-\; \tilde{g} Z_\mu J^\mu_Z\,. \end{aligned} \tag{S.17}$$

Now, let us focus on the low-energy amplitudes without incoming or outgoing vector particles. The internal lines of the Feynman diagrams do include the vector propagators, but in the low-energy regime we may approximate such propagators as

$$\frac{-ig^{\mu\nu}}{q^2 - M^2} \;\approx\; \frac{+ig^{\mu\mu}}{M^2}\,. \tag{S.18}$$

This approximation often fails in loop graphs, but it works well at the tree level when all the momenta are small compared to M_W or M_Z.

In Lagrangian terms, the approximation (S.18) corresponds to neglecting the kinetic terms for the vector fields compared to the mass terms,

$$\mathcal{L}^{\rm weak} \;\approx\; M_W^2 W^+_\mu W^{-,\mu} \;+\; \tfrac{1}{2}M_Z^2 Z_\mu Z^\mu \;-\; \frac{g}{\sqrt{2}}\left(W^+_\mu J^{-,\mu}_L \;+\; W^-_\mu J^{+,\mu}_L\right) \;-\; \tilde{g} Z_\mu J^\mu_Z\,. \tag{S.19}$$

In this formula, the vector fields appear as auxiliary fields with algebraic (*i.e.*, derivative-less) equations of motions. Consequently, we may solve those equations of motions and plug them back into eq. (S.19); the result is an effective current-current Lagrangian

$$\mathcal{L}^{\rm eff} \;=\; -\frac{g^2}{2M_W^2}\, J^{+,\mu}_L J^-_{L,\mu} \;-\; \frac{\tilde{g}^2}{M_Z^2}\, J^\mu_Z J_{Z,\mu} \tag{S.20}$$

governing the low-energy weak interactions of quarks and leptons. For historic reasons, this effective Lagrangian is usually written as

$$\mathcal{L}^{\text{eff}} \;=\; -\frac{4G_F}{\sqrt{2}} \left[J_L^{+,\mu} J_{L,\mu}^{-} \;+\; \rho J_Z^{\mu} J_{Z,\mu} \right] \tag{1}$$

where

$$G_F \;=\; \frac{\sqrt{2} g^2}{8 M_W^2} \tag{S.21}$$

is the original Fermi's weak coupling constant, and

$$\rho \;=\; \frac{\tilde{g}^2}{M_Z^2} \Big/ \frac{g^2}{M_W^2} \;=\; \frac{M_W^2}{M_Z^2} \times \frac{1}{\cos^2\theta} \tag{S.22}$$

is the ratio of neutral-current to charged-current weak interactions.

In the standard model,

$$G_F \;=\; \frac{\sqrt{2}}{2 v_h^2} \tag{S.23}$$

and $\rho = 1$ because $M_W = M_Z \times \cos\theta$. Adding the triplet VEV increases the W mass but does not affect the Z mass and the mixing angle θ. Consequently, the Fermi coupling becomes

$$G_F \;=\; \frac{\sqrt{2}}{2 v_h^2 + 8 v_\varphi^2}\,, \tag{S.24}$$

and the ρ parameter becomes

$$\rho \;=\; \frac{M_W^2}{M_Z^2 \cos^2\theta} \;=\; \frac{v_h^2 + 4 v_\varphi^2}{v_h^2} \;>\; 1. \tag{S.25}$$