1. Consider an $SU(N) \times SU(N)$ supersymmetric gauge theory with the following chiral superfields:

\[ Q_f \in (1, N) \quad (f = 1, \ldots, N), \]
\[ \Omega \in (N, \overline{N}) \quad \text{(just one)}, \]
\[ \tilde{Q}_f \in (\overline{N}, 1) \quad (f = 1, \ldots, N). \]  

For $W_{\text{tree}} = 0$, this theory has a moduli space parametrized by

\[ M_{ff'} = \tilde{Q}_f \Omega Q_{f'}, \quad B = \det(Q), \quad \tilde{B} = \det(\tilde{Q}), \quad C = \det(\Omega) \]

subject to a classical constraint

\[ \det(M) = \tilde{B} \times C \times B. \]  

(*) Before doing anything else, write down the anomaly-free flavor symmetries of the theory.

(a) Argue that the non-perturbative quantum effects do not generate an effective superpotential for the moduli but instead modify the moduli space geometry according to

\[ \det(M) = \tilde{B} \times C \times B + \Lambda_1^{2N} \times B + \tilde{B} \times \Lambda_2^{2N}. \]  

(b) Despite quantum corrections (4), the theory has a vacuum $\langle M \rangle = \langle B \rangle = \langle \tilde{B} \rangle = \langle C \rangle = 0$ where all flavor symmetries remain unbroken. Check ’t Hooft’s anomaly matching condition for this vacuum.

Hint: Quantum corrections (4) give mass to one combination of $B$ and $\tilde{B}$ fields.
2. The second problem is about Klebanov–Strassler cascade. Let’s start with a generic question.

(a) Consider a generic SUSY gauge theory with a non-trivial infrared fixed point. Let’s perturb the theory’s Lagrangian by

$$\Delta \mathcal{L} = \int d^2 \theta \lambda \hat{O} + \text{H.c.}$$  \hspace{1cm} (5)

where $\lambda$ is a small coupling and $\hat{O}(x)$ is a chiral gauge-invariant operator of $R$-charge $r$. Show that in the IR limit of the theory, the perturbation (5) is: irrelevant for $r > 2$, marginal for $r = 2$, and relevant for $r < 2$.

The Klebanov–Strassler model is an $SU(A) \times SU(B)$ SUSY gauge theory with chiral superfields $Q_1, Q_2 \in (A, B)$ and $\tilde{Q}_1, \tilde{Q}_2 \in (\overline{A}, B)$, and a quartic superpotential

$$W_{\text{tree}} = \lambda \text{tr}(\tilde{Q}_1 Q_1 \tilde{Q}_2 Q_2 - \tilde{Q}_1 Q_2 \tilde{Q}_2 Q_1).$$  \hspace{1cm} (6)

Let’s simplify the model by turning off the $SU(B)$ gauge coupling (i.e., setting $g_B = 0$), which turns the theory into SQCD (with $N_c = A$ and $N_F = 2B$) perturbed by the superpotential (6).

(b) What are the limits on flavor/color ratio of SQCD with a non-trivial infrared fixed point?

(c) Let’s go to such a fixed point and turn on a small $\lambda \ll (1/E)$ as a perturbation. Show that this perturbation is irrelevant for $B > A$, marginal for $B = A$, and relevant for $B < A$. Hint: use (a).

Let’s focus on the relevant case for $B < A$. A relevant perturbation becomes more important at lower energies, until eventually it becomes important in the very low energy limit of the theory. The effect of a relevant (6) perturbation becomes clear in the Seiberg dual of the un-perturbed SQCD:

(d) Show that Seiberg duality turns this perturbation into mass terms for the gauge-singlet $M_{f'f'}$ fields.
(e) Integrate out the massive $M_{ff}$ fields from the very-low-energy theory, and show that this generates a quartic superpotential for the dual quarks which looks exactly like (6) (albeit the overall coefficient may be different, $\lambda^{\text{dual}} \neq \lambda$). Also show that in the dual theory, this quartic superpotential is irrelevant because $A^{\text{dual}} < B^{\text{dual}}$.

Now let’s momentarily set $\lambda = 0$ but instead turn on the $SU(B)$ gauge coupling, $g_B \neq 0$.

(f) Write down the Shifman–Vainshtein $\beta$–functions for the two gauge couplings, and show that for $A > B > \frac{3}{4}A$, the $g_A$ has a non-trivial infrared fixed point while the $g_B$ is infrared-free.

Note: Although $\beta_B < 0$ when both $g_A$ and $g_B$ are weak, $\beta_B$ becomes positive when $g_A$ flows to its fixed point.

And now let’s turn on all three couplings, $g_A, g_B, \lambda \neq 0$. Along the renormalization flow in the IR direction, $g_A$ grows until it reaches its fixed point, $g_B$ grows at first but then becomes weak again, while $\lambda$ — or rather dimensionless $\lambda \times E$ — first becomes weak but eventually grows strong again. And when it does become strong, we go to the Seiberg dual of the $SU(A)$ theory — which looks exactly like the original theory, but with $B' > A'$.

And then the process repeats itself, with $g_A$ and $g_B$ reversing roles: $g_B$ grows from weak coupling to a fixed point, $g_A$ becomes weak, while $\lambda E$ first becomes weak but eventually grows strong, — and then we go to the Seiberg dual of the $SU(B')$, — which gives us back the original theory with $A'' > B''$. And then we have another cascade, etc., etc..

Note that at each step $B' = B$ while $A' = 2B - A$, hence $|A - B|$ remains constant while $A + B$ is reduced at each step. Eventually, we end up with $A \geq 2B$ (or $B \geq 2A$) and the cascade stops.

(g) Explain how this duality cascade works in terms of (c), (e), and (f).

(h) Finally, let’s start the cascade with $A = N \times (k + 1)$ and $B = N \times k$. Show that the cascade ends with an SYM theory ($G = SU(N)$) and hence gaugino condensation and confinement.