PHY–396 K. Mid-term Exam. Due Thursday, November 6, 2008.

Please do not waste time and paper by copying the posted homework solutions. If you need to use any homework result, simply reference the appropriate question or equation and go ahead. Ditto for anything explicitly derived in class.

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1. Consider a theory of N Dirac spinor fields Ψ^i of the same mass m,

$$\mathcal{L} = \sum_{i=1}^{N} \overline{\Psi}_i (i\partial\!\!\!/ - m)\Psi^i. \tag{1}$$

This theory has an obvious $U(N)$ global symmetry which acts as $\Psi^i(x) \to U^i{}_j \Psi^j(x)$.

(a) Show that the Noether currents of this symmetry form an $N \times N$ matrix

$$J_i^{\mu j} = \overline{\Psi}_i \gamma^\mu \Psi^j. \tag{2}$$

(b) Verify that all these currents are conserved by equations of motion of the Ψ^j and $\overline{\Psi}_i$ fields.

(c) In the quantum theory, all currents and charges become operators in the fermionic Fock space. Expand the global charges

$$\hat{Q}_i^{\,j} = \int d^3\mathbf{x}\, \hat{J}_i^{0j}(\mathbf{x}) \tag{3}$$

in terms of fermionic creation and annihilation operators. Subtract the charges of the physical vacuum.

(d) Verify that all these charges commute with the Hamiltonian of the theory.

(e) Verify that the charges (3) satisfy the $U(N)$ commutations relations

$$\left[\hat{Q}_i^{\,j}, \hat{Q}_k^{\,\ell}\right] = \delta_k^j \hat{Q}_i^{\,\ell} - \delta_i^\ell \hat{Q}_k^{\,j}. \tag{4}$$

2. Consider a non-abelian $SO(3)$ gauge theory coupled to a triplet of *complex* scalar fields Φ_a,

$$\mathcal{L} \;=\; -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \;+\; D_\mu \Phi^*_a \, D^\mu \Phi_a \;-\; V, \tag{5}$$

$$V \;=\; m^2 \, \Phi^*_a \Phi_a \;+\; \alpha \left(\Phi^*_a \Phi_a\right)^2 \;+\; \beta \sum_a \left|\epsilon_{abc} \Phi^*_b \Phi_c\right|^2 , \tag{6}$$

where indices $a, b, c = 1, 2, 3$ and repeated indices are summed over.

(a) Identify all symmetries of this theory, global or local, discrete or continuous. For simplicity, skip the spacetime symmetries (Lorentz, translations, P, and T) and focus on the internal symmetries only.

(b) Show that for $\alpha, \beta > 0$ but $m^2 < 0$ all minima of the scalar potential (6) are related by symmetries to

$$\langle \Phi \rangle \;=\; \frac{v}{\sqrt{2}} \times (1, 0, 0), \qquad v \;=\; \sqrt{\frac{-m^2}{\alpha}}\,. \tag{7}$$

(c) Which symmetries are spontaneously broken by this vacuum expectation value (VEV)? Which symmetries remain unbroken? How many massless Goldstone bosons does this symmetry breaking calls for and what should be their quantum numbers? Which vector fields becomes massive by the Higgs mechanism and which remain massless?

(d) Expand the Lagrangian (5) in powers of $\Phi_a(x) - \left\langle \Phi_a \right\rangle$ and $A^a_\mu(x)$, write down the quadratic terms, and use them to determine the mass spectrum of the theory. Check that this spectrum agrees with predictions you have made in part (c).

3. In three spacetime dimensions (two space plus one time) an antisymmetric Lorentz tensor $F^{\mu\nu} = -F^{\nu\mu}$ is equivalent to an axial Lorentz vector, $F^{\mu\nu} = \epsilon^{\mu\nu\lambda} F_\lambda$. Consequently, in 3D one can have a massive photon despite unbroken gauge invariance of the electromagnetic field $A_\mu(x)$. Indeed, consider the following Lagrangian:

$$\mathcal{L} \;=\; -\frac{1}{2} F_\lambda F^\lambda \;+\; \frac{m}{2}\, F_\lambda A^\lambda \tag{8}$$

where

$$F^\lambda(x) \;=\; \tfrac{1}{2} \epsilon^{\lambda\mu\nu} F_{\mu\nu}(x) \;=\; \epsilon^{\lambda\mu\nu} \partial_\mu A_\nu(x). \tag{9}$$

(a) Show that the action $S = \int d^3x \mathcal{L}$ is gauge invariant (although the Lagrangian (8) is not invariant).

(b) Write down equations of motion (including Jacobi identities) for the $F_\lambda(x)$ fields.

(c) Show that these equations imply Klein–Gordon equations $(\partial^2 + m^2) F_\lambda(x) = 0$ for all components of the F field.

(d) Write down plane-wave solutions of the field equations. Count the gauge-invariant independent modes for each $p^\mu = (E_\mathbf{p}, \mathbf{p})$ and show that there is only one physical polarization. Explain what this means for the massive photons in 3D (but don't go through the gory details of quantizing the A^μ field). Find out the specific $SO(2)$ "spin" state of the massive photon from the polarization of the $\mathbf{p} = 0$ mode.

(e) Write down Noether stress-energy tensor for the theory in question, then add a suitable $\epsilon^{\mu\kappa\lambda} \partial_\kappa \mathcal{K}^\nu_\lambda$ term to make $T^{\mu\nu}$ symmetric and gauge invariant.
Hints: (1) See homework sets 1 and 2 for 4D examples of a $\mathcal{K}$ correction to the $T^{\mu\nu}_{\text{Noether}}$.
(2) Your result should have a simple form in terms of the F_λ.

The non-abelian gauge fields in 3D may also be massive despite unbroken gauge symmetry. In matrix notations, the Lagrangian is

$$\mathcal{L} \;=\; -\frac{1}{g^2}\, \mathrm{tr}\big(F^\lambda F_\lambda\big) \;+\; \frac{k}{4\pi}\, \epsilon^{\lambda\mu\nu}\, \mathrm{tr}\left(A_\lambda\, \partial_\mu A_\nu \;+\; \frac{2i}{3}\, A_\lambda A_\mu A_\nu \right) \tag{10}$$

where $F^\lambda \;=\; \frac{1}{2} \epsilon^{\lambda\mu\nu} F^{\text{non-abelian}}_{\mu\nu}$. The second term in (10) is called *the Chern–Simons term,* and its coefficient k — called the *the Chern–Simons level* — should be an integer; k can be positive, zero, or negative, but for this test we assume $k > 0$.

(f) Show that the action $S = \int d^3x\mathcal{L}$ is invariant under *infinitesimal* gauge transforms of the non-abelian vector field.

($\star$) *For extra credit,* show that for finite gauge transformations $U(x)$, the action changes by a field-independent amount

$$\Delta S \;=\; \frac{k}{6\pi}\int d^3x\,\epsilon^{\lambda\mu\nu}\,\mathrm{tr}\Big(U^\dagger(\partial_\lambda U)U^\dagger(\partial_\mu U)U^\dagger(\partial_\nu U)\Big)\,. \tag{11}$$

For your information (but don't try to prove this yourself for the test): The integrand here is a total derivative (although it's rather hard to see), but the integral may be non-zero for a topologically non-trivial $U(x)$.

Note that it's OK for a classical symmetry to change the action S *by a constant amount* (*i.e.*, ΔS does not depend on any fields, only on the symmetry parameters such as $U(x)$). In the quantum theory, the invariance of the path integral of e^{iS} imposes a stronger requirement: for any symmetry, $\Delta S = 2\pi \times$ an integer.

For the theory at hand, eq. (11) yields $\Delta S = 2\pi k \times$ an integer for any $U(x)$. (Don't try to prove this during the test.) This means that the gauge symmetry is OK at the quantum level as long as the Chern–Simons level k of the theory is an integer.

(g) Now, write down field equations for the non-abelian $F_\lambda(x)$ in a manifestly covariant form. Compare those equations to the abelian case (part (b)) and determine the mass of the vector particles.

(h) Finally, show that these field equations imply

$$(D^2+m^2)F^\lambda \;=\; i\epsilon^{\lambda\mu\nu}[F_\mu,F_\nu] \;=\; 2i\epsilon^{\lambda\mu\nu}F_\mu F_\nu\,. \tag{12}$$