BRST Symmetry of QCD

The BRST symmetry — named after Becchi, Rouet, Stora, and Tyutin who discovered it — relates the ghosts and the longitudinal gluons to each other and makes sure that the un-physical quanta of the c^a, $\bar{c}^a$, and $\partial^\mu A^a_\mu$ fields always cancel each other out from all the physical processes.

Before we spell out the action of the BRST symmetry, let's write the QCD Lagrangian as

$$\mathcal{L} \;=\; \mathcal{L}_{\text{classical}} \;+\; \mathcal{L}^{\text{gauge fixing}}_{\text{and ghosts}}\,, \tag{1}$$

$$\mathcal{L}_{\text{classical}} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; \sum_f \overline{\Psi}_{fi}(i\not{D}+m_f)\Psi^{fi}, \tag{2}$$

$$\mathcal{L}^{\text{gauge fixing}}_{\text{and ghosts}} \;=\; \partial_\mu \bar{c}^c D_\mu c^a \;+\; \tfrac{1}{2}\xi(b^a)^2 \;-\; b^a\partial^\mu A^a_\mu\,. \tag{3}$$

The $b^a(x)$ are auxiliary fields: for $\xi = 0$ they are Lagrange multipliers for the Landau gauge condition $\partial^\mu A^a_\mu = 0$, while for $\xi \neq 0$ we may eliminate the b^a by their equations of motion $\xi b^a = \partial^\mu A^a_\mu$ which brings the gauge-fixing terms to their standard form $-(1/2\xi)(\partial^\mu A^a_\mu)^2$. Let's also use matrix notations for all the adjoint fields,

$$\mathcal{A}_\mu \;=\; g\sum_a A^a_\mu T^a, \qquad \mathcal{C} \;=\; g\sum_a c^a T^a, \qquad \overline{\mathcal{C}} \;=\; g\sum_a \bar{c}^a T^a, \qquad \mathcal{B} \;=\; g\sum_a b^a T^a, \tag{4}$$

where we have multiplied all the matrix-valued fields by the gauge coupling g so that the gauge and the BRST symmetries act in a g-independent manner. Thus,

$$\begin{aligned} D_\mu\Psi(x) \;&=\; \partial_\mu\Psi(x) \;+\; i\mathcal{A}_\mu(x)\Psi(x),\\ D_\mu\mathcal{C}(c) \;&=\; \partial_\mu\mathcal{C}(x) \;+\; i\{\mathcal{A}_\mu(x),\mathcal{C}(x)\},\\ \mathcal{F}_{\mu,\nu}(x) \;&=\; \partial_\mu\mathcal{A}_\nu(x) \;-\; \partial_\nu\mathcal{A}_\mu(x) \;+\; i[\mathcal{A}_\mu(x),\mathcal{A}_\nu(x)], \end{aligned} \tag{5}$$

under infinitesimal gauge transform $U(x) = 1 + i\Lambda(x)$

$$\delta\Psi = i\Lambda\Psi, \quad \delta\overline{\Psi} = -i\Psi\Lambda, \quad \delta\mathcal{A}_\mu = -D_\mu\Lambda, \quad \delta\mathcal{C} = i[\Lambda,\mathcal{C}], \quad \delta\overline{\mathcal{C}} = i[\Lambda,\overline{\mathcal{C}}], \quad \delta\mathcal{B} = i[\Lambda,\mathcal{B}], \tag{6}$$

and

$$\mathcal{L}_{\rm cl} \;=\; \frac{-1}{2g^2}\,\mathrm{tr}\,(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}) \;+\; \sum_f \overline{\Psi}_f(i\not{D}+m)\Psi^f, \tag{7}$$

$$\mathcal{L}_{\rm gf+gh} \;=\; \frac{2}{g^2}\,\mathrm{tr}\left(\partial_\mu\overline{\mathcal{C}}D^\mu\mathcal{C} \;+\; \frac{\xi}{2}\mathcal{B}^2 \;-\; \mathcal{B}\partial^\mu\mathcal{A}_\mu\right). \tag{8}$$

In matrix notations,[⋆] the BRST symmetry acts as

$$\delta\Psi(x) \;=\; \epsilon\{Q,\Psi(x)\} \;=\; -\epsilon\,\mathcal{C}(x)\Psi(x), \tag{10}$$

$$\delta\overline{\Psi}(x) \;=\; \epsilon\{Q,\overline{\Psi}(x)\} \;=\; -\epsilon\,\overline{\Psi}(x)\mathcal{C}(x), \tag{11}$$

$$\delta\mathcal{A}_\mu(x) \;=\; \epsilon[Q,\mathcal{A}_\mu(x)] \;=\; i\epsilon D_\mu\mathcal{C}(x), \tag{12}$$

$$\delta\mathcal{C}(x) \;=\; \epsilon\{Q,\mathcal{C}(x)\} \;=\; \epsilon\,\mathcal{C}(x)\mathcal{C}(x), \tag{13}$$ [†]

$$\delta\overline{\mathcal{C}}(x) \;=\; \epsilon\{Q,\overline{\mathcal{C}}(x)\} \;=\; -i\epsilon\,\mathcal{B}(x), \tag{14}$$

$$\delta\mathcal{B}(x) \;=\; \epsilon[Q,\mathcal{B}(x)] \;=\; 0, \tag{15}$$

where ϵ is an 'infinitesimal' Grassmann number and Q is the fermionic operator generating the BRST symmetry. Note that ϵ is x independent, so the BRST symmetry is global. Also note that although the BRST symmetry is fermionic, it has nothing to do with the supersymmetry.

⋆ For your reference, in component notations the BRST operator Q acts as

$$\begin{aligned}
\{Q,\Psi^i\} \;&=\; -gc^a(T^a)^i{}_j\Psi^j, &\qquad \{Q\,\overline{\Psi}_i\} \;&=\; -g\overline{\Psi}_j(T^a)^j{}_ic^a,\\
[Q,A^a_\mu] \;&=\; i\partial_\mu c^a \;-\; igf^{abc}A^b_\mu c^c, &\qquad \{Q,c^a\} \;&=\; \frac{ig}{2}\,f^{abc}c^bc^c,\\
\{Q,\bar{c}^a\} \;&=\; -ib^a, &\qquad [Q,b^a] \;&=\; 0.
\end{aligned} \tag{9}$$

† Despite Fermi statistics of the ghost fields, $\mathcal{C}\mathcal{C}$ does not vanish because $\mathcal{C}$ is a whole matrix of fermionic fields. Consequently,

$$\mathcal{C}\mathcal{C} \;=\; T^aT^bc^ac^b \;=\; -T^bT^ac^ac^b \;=\; \tfrac{1}{2}[T^a,T^b]\times c^ac^b \;=\; \tfrac{i}{2}f^{abc}c^ac^bT^c. \tag{16}$$

As far as quark and gluon fields are concerned, the BRST transforms (10), (11), and (12) look like infinitesimal gauge transforms with gauge parameter

$$\Lambda(x) \;=\; i\epsilon\,\mathcal{C}(x). \tag{17}$$

This immediately tells us that the classical QCD Lagrangian (7) is not only gauge invariant but also BRST invariant. The gauge fixing and ghost terms (8) in the quantum Lagrangian are not gauge invariant, so proving their BRST symmetry is more complicated. But before I do that, let me address another issue, the nilpotency of the BRST operator Q.

Theorem: *The BRST operator Q is nilpotent, $Q^2 = 0$.*
To prove this theorem, we need to show that $Q^2 = \frac{1}{2}\{Q, Q\}$ commutes with all the fields; by Jacobi identity, this is equivalent to

$$\{Q, [Q, \text{any bosonic field}]\} \;=\; 0 \qquad \text{and} \qquad [Q, \{Q, \text{any fermionic field}\}] \;=\; 0. \tag{18}$$

Let's verify these double-commutator formulae field by field. Obviously

$$\{Q, [Q, \mathcal{B}]\} \;=\; 0 \qquad \text{and} \qquad [Q, \{Q, \overline{\mathcal{C}}\}] \;=\; -i[Q, \mathcal{B}] \;=\; 0. \tag{19}$$

Less obviously but still rather simply

$$[Q, \{Q, \Psi\}] \;=\; -[Q, \mathcal{C}\Psi] \;=\; -\{Q, C\}\Psi \;+\; \mathcal{C}\{Q, \Psi\} \;=\; -\mathcal{C}\mathcal{C}\Psi \;+\; \mathcal{C}\mathcal{C}\Psi \;=\; 0 \tag{20}$$

and similarly $[Q, \{Q, \overline{\Psi}\}] = 0$. Likewise

$$[Q, \{Q, \mathcal{C}\}] \;=\; [Q, \mathcal{C}\mathcal{C}] \;=\; \{Q, \mathcal{C}\}\mathcal{C} \;-\; \mathcal{C}\{Q, \mathcal{C}\} \;=\; \mathcal{C}\mathcal{C}\mathcal{C} \;-\; \mathcal{C}\mathcal{C}\mathcal{C} \;=\; 0. \tag{21}$$

Finally, the gauge field $\mathcal{A}_\mu$ takes a bit of algebra:

$$\begin{aligned}
\{Q, [Q, \mathcal{A}_\mu]\} \;&=\; i\{Q, D_\mu \mathcal{C}\} \\
&=\; i[Q, D_\mu]\mathcal{C} \;+\; iD_\mu\{Q, \mathcal{C}\} \\
&=\; -\{[Q, \mathcal{A}_\mu], \mathcal{C}\} \;+\; iD_\mu(\mathcal{C}\mathcal{C}) \\
&=\; -i\{D_\mu \mathcal{C}, \mathcal{C}\} \;+\; i\{D_\mu \mathcal{C}, \mathcal{C}\} \\
&=\; 0.
\end{aligned} \tag{22}$$

Quod erat demonstrandum.

Having proved the nilpotency of the BRST operator, the simplest way to establish that the ghost and gauge fixing terms in QCD Lagrangian are BRST symmetric is to show that

$$\mathcal{L}_{\text{gf+gh}} \;=\; \{Q, Z\} \tag{23}$$

for some fermionic operator Z. Indeed, given eq. (23), we immediately have

$$\left[Q, \mathcal{L}_{\text{gf+gh}}\right] \;=\; [Q, \{Q, Z\}] \;=\; [Q^2, Z] \;=\; 0 \quad \text{by nilpotency of } Q. \tag{24}$$

To verify eq. (23), we take

$$Z \;=\; \frac{2i}{g^2}\,\operatorname{tr}\left(\overline{\mathcal{C}} \times \left(\frac{\xi}{2}\mathcal{B} \,-\, \partial^\mu \mathcal{A}_\mu\right)\right). \tag{25}$$

Anticommuting this operator with Q we obtain

$$\begin{aligned}
\{Q, Z\} \;&=\; \frac{2i}{g^2}\,\operatorname{tr}\left(\{Q, \overline{\mathcal{C}}\} \times \left(\frac{\xi}{2}\mathcal{B} \,-\, \partial^\mu \mathcal{A}_\mu\right) \,-\, \overline{\mathcal{C}} \times \left[Q, \left(\frac{\xi}{2}\mathcal{B} \,-\, \partial^\mu \mathcal{A}_\mu\right)\right]\right) \\
&=\; \frac{2i}{g^2}\,\operatorname{tr}\left(-i\mathcal{B} \times \left(\frac{\xi}{2}\mathcal{B} \,-\, \partial^\mu \mathcal{A}_\mu\right) \,-\, \overline{\mathcal{C}} \times (0 \,-\, i\partial^\mu D_\mu \mathcal{C})\right) \\
&=\; \frac{2}{g^2}\,\operatorname{tr}\left(\frac{\xi}{2}\mathcal{B}^2 \,-\, \mathcal{B}\partial^\mu \mathcal{A}_\mu \,- \overline{\mathcal{C}}\partial^\mu D_\mu \mathcal{C}\right) \\
&=\; \mathcal{L}_{\text{gf+gh}} \qquad \text{(up to a total derivative)},
\end{aligned} \tag{26}$$

which proves the BRST symmetry of the gauge-fixing and ghost parts of the QCD action. And as I have argued a couple of pages above, the classical QCD action is BRST symmetric because of its gauge invariance.

Because BRST is an exact symmetry of the fully-quantized non-abelian gauge theory, it leads to a variety of Ward–Takahashi-like identities for various amplitudes. Using these identities one can show that all the UV divergences of the theory can be cancelled by the counterterms which are present in the bare Lagrangian. In other words, we do not need counterterms for the gluon mass, or the ghost mass, or any non-gauge-invariant combinations of the gluon fields. Thus, *thanks to the BRST symmetry, the non-abelian gauge theories are renormalizable.*