1. Continuing the previous homework set, consider a classical theory made of a complex scalar field $\Phi$ of charge $q \neq 0$ and the the EM fields:

$$L_{\text{net}} = D^\mu \Phi^* D_\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}$$  \hspace{1cm} (1)

where

$$D_\mu \Phi = (\partial_\mu + i q A_\mu) \Phi \quad \text{and} \quad D_\mu \Phi^* = (\partial_\mu - i q A_\mu) \Phi^*$$  \hspace{1cm} (2)

are the covariant derivatives.

(a) Write down the equation of motion for all fields in a covariant form. Also, write down the electric current

$$J^\mu \overset{\text{def}}{=} - \frac{\partial L}{\partial A_\mu}$$  \hspace{1cm} (3)

in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

(b) Write down the Noether stress-energy tensor for the whole field system and show that

$$T^{\mu \nu}_{\text{net}} \equiv T^{\mu \nu}_{\text{EM}} + T^{\mu \nu}_{\text{mat}} = T^{\mu \nu}_{\text{Noether}} + \partial_\lambda K^{\lambda \mu \nu},$$  \hspace{1cm} (4)

where

$$T^{\mu \nu}_{\text{EM}} = - F^{\mu \alpha} F_{\alpha \nu} + \frac{1}{4} g^{\mu \nu} F_{\alpha \beta} F^{\alpha \beta}$$  \hspace{1cm} (5)

as for the free EM,

$$K^{\lambda \mu \nu} \equiv - K^{\mu \lambda \nu} = - F^{\lambda \mu} A_\nu,$$  \hspace{1cm} (6)

also exactly as for the free EM, and

$$T^{\mu \nu}_{\text{mat}} = D^\mu \Phi^* D^\nu \Phi + D^\nu \Phi^* D^\mu \Phi - g^{\mu \nu} (D_\lambda \Phi^* D^\lambda \Phi - m^2 \Phi^* \Phi).$$  \hspace{1cm} (7)

Note: In the presence of an electric current $J^\mu$, the $\partial_\lambda K^{\lambda \mu \nu}$ correction to the electromagnetic stress-energy tensor contains an extra $J^\mu A_\nu$ term. This term is important for obtaining a gauge-invariant stress-energy tensor (7) for the scalar field.
(c) Use the scalar fields’ equations of motion and the non-commutativity of covariant derivatives

\[ [D_\mu, D_\nu] \Phi = iq F_{\mu\nu} \Phi, \quad [D_\mu, D_\nu] \Phi^* = -iq F_{\mu\nu} \Phi^* \quad (8) \]

to show that

\[ \partial_\mu T^{\mu\nu}_{\text{mat}} = + F^{\nu\lambda} J_\lambda \quad (9) \]

and therefore the net stress-energy tensor (4) is conserved.

Note: the last statement follows from problem 1.3 (e). Do not redo that problem here, just quote the result.

2. In class, we discussed field multiplets \( \Psi_i(x) \) which transform as (complex) vectors under the \( SU(N) \) symmetry,

\[ \Psi^I(x) = U(x) \Psi(x) \quad i.e. \quad \Psi^I_i(x) = \sum_j U^I_j(x) \Psi_j(x), \quad i, j = 1, 2, \ldots, N \quad (10) \]

where \( U(x) \) is an \( x \)-dependent unitary \( N \times N \) matrix, \( \det U(x) \equiv 1 \). Now consider the adjoint multiplet \( \Phi^{ij}(x) \) of fields: for each \( x \), it comprises a traceless hermitian \( N \times N \) matrix \( \Phi(x) \) which transforms according to

\[ \Phi^I(x) = U(x) \Phi(x) U^I(x), \quad i.e. \quad \Phi^{ij}_I(x) = \sum_{k, \ell} U^I_i(x) \Phi_{k}(x) U^\dagger_{\ell j}(x). \quad (11) \]

Note that this transformation law preserves the \( \Phi^I = \Phi \) and \( \text{tr}(\Phi) = 0 \) conditions.

The covariant derivative acts on the adjoint multiplet according to

\[ D_\mu \Phi(x) = \partial_\mu \Phi(x) + i[A_\mu(x), \Phi(x)] \equiv \partial_\mu \Phi(x) + iA_\mu(x) \Phi(x) - i\Phi(x) A_\mu(x) \quad (12) \]

(a) Verify that this derivative is indeed covariant and \( D_\mu \Phi(x) \) transforms under the local \( SU(N) \) symmetry exactly like \( \Phi(x) \) itself.

(b) Show that \( [D_\mu, D_\nu] \Phi(x) = i[F_{\mu\nu}(x), \Phi(x)] \).
The non-abelian tension field $F_{\mu\nu}(x)$ itself transforms according to the adjoint representation of the local symmetry, $F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^\dagger(x)$. Hence, the covariant derivative acts on the tension field according to $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + i[A_\lambda, F_{\mu\nu}]$.

(c) Verify the non-abelian Bianchi identity $D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0$.

(d) Show that for an infinitesimal variation of the non-abelian gauge field $A_\nu(x) \rightarrow A_\nu(x) + \delta A_\nu(x)$, the tension varies according to $\delta F_{\mu\nu}(x) = D_\mu \delta A_\nu(x) - D_\nu \delta A_\mu(x)$.

(e) Note: This question was changed Friday 9/12 at 22:30.

The Yang–Mills theory is has a non-abelian gauge symmetry, but its only fields are the gauge fields $A_i^{\mu j}(x) = \sum_a (\chi_a^{\lambda})^{i j} \times A_a^{\mu}(x)$ themselves; there are no other fields. The Yang–Mills Lagrangian is

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) = \sum_a -\frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu}. \quad (13)$$

Write down the classical equations of motion for this theory.