1. Consider a non-abelian gauge theory comprising \( N \) complex scalar fields \( \phi_i(x) \) and \( N^2 - 1 \) real vector fields \( A^a_\mu(x) \). In matrix notations, the Lagrangian of the theory is

\[
\mathcal{L} = -\frac{1}{2g^2} \text{tr}(F^{\mu\nu} F_{\mu\nu}) + D_\mu \Phi^\dagger D^\mu \Phi - m^2 \Phi^\dagger \Phi. \tag{1}
\]

(a) Derive classical equations of motion for all the fields and write those equation in a covariant form. In particular, show that the vector fields satisfy

\[
D_\mu F^{\mu\nu}(x) = g^2 J^\nu(x) \equiv g^2 \sum_a \frac{\lambda^a}{2} \times J^{a\mu}(x) \tag{2}
\]

where the currents \( J^{a\mu}(x) \) involve the scalar fields and their covariant derivatives.

(b) Show that regardless of the specific form of the currents \( J^{a\mu}(x) \), eqs. (2) require those currents to be covariantly conserved, \( D_\mu J^{\mu}(x) = 0 \).

(c) Show that when the scalar fields \( \Phi(x) \) and \( \Phi^\dagger(x) \) satisfy their equations of motion, the currents \( J^{\mu}(x) \) are indeed covariantly conserved.

Note that covariantly conserved currents \( J^{a\mu}(x) \) do not give rise to conserved charges \( Q^a = \int d^3 x \cdot J^{a0}(x, t) \). That’s one more reason why the non-abelian gauge theories are much more complicated than the electromagnetism.

2. Consider a massive relativistic vector field \( A^\mu(x) \) with the Lagrangian density

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - A^\mu J_\mu \tag{3}
\]

(in \( \hbar = c = 1 \) units) where the current \( J^\mu(x) \) is a fixed source for the \( A^\mu(x) \) field. Because of the mass term, the Lagrangian (3) is not gauge invariant. However, we assume that the current \( J^\mu(x) \) is conserved, \( \partial_\mu J^\mu(x) = 0 \).
In an earlier homework (set 1, problem 1) we have derived the Euler–Lagrange equations for the massive vector field. In this problem, we develop the Hamiltonian formalism for the \(A^\mu(x)\). Our first step is to identify the canonically conjugate “momentum” fields.

(a) Show that \(\partial L / \partial \dot{A} = -E\) but \(\partial L / \partial \dot{A}_0 \equiv 0\).

In other words, the canonically conjugate field to \(A(x)\) is \(-E(x)\) but the \(A_0(x)\) does not have a canonical conjugate! Consequently,

\[
H = \int d^3x \left( -\dot{A}(x) \cdot E(x) - L \right).
\]

(b) Show that in terms of the \(A, E,\) and \(A_0\) fields, and their space derivatives,

\[
H = \int d^3x \left\{ \frac{1}{2} E^2 + A_0 (J_0 - \nabla \cdot E) - \frac{1}{2} m^2 A_0^2 + \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2} m^2 A^2 - J \cdot A \right\}.
\]

Because the \(A_0\) field does not have a canonical conjugate, the Hamiltonian formalism does not produce an equation for the time-dependence of this field. Instead, it gives us a time-independent equation relating the \(A_0(x, t)\) to the values of other fields at the same time \(t\).

Specifically, we have

\[
\frac{\delta H}{\delta A_0(x)} \equiv \frac{\partial H}{\partial A_0} \bigg|_x - \nabla \cdot \frac{\partial H}{\partial (\nabla A_0)} \bigg|_x = 0.
\]

At the same time, the vector fields \(A\) and \(E\) satisfy the Hamiltonian equations of motion,

\[
\frac{\partial}{\partial t} A(x, t) = - \frac{\delta H}{\delta E(x)} \bigg|_t \equiv - \left[ \frac{\partial H}{\partial E} - \nabla_i \frac{\partial H}{\partial (\nabla_i E)} \right]_{(x,t)},
\]

\[
\frac{\partial}{\partial t} E(x, t) = + \frac{\delta H}{\delta A(x)} \bigg|_t \equiv + \left[ \frac{\partial H}{\partial A} - \nabla_i \frac{\partial H}{\partial (\nabla_i A)} \right]_{(x,t)}.
\]

(c) Write down the explicit form of all these equations.

(d) Verify that the equations you have just written down are equivalent to the relativistic Euler–Lagrange equations for the \(A^\mu(x)\), namely

\[
(\partial^\mu \partial_\mu + m^2)A^\nu = \partial^\nu (\partial_\mu A^\mu) + J^\nu
\]

and hence \(\partial_\mu A^\mu(x) = 0\) and \((\partial^\nu \partial_\nu + m^2)A^\mu = 0\) when \(\partial_\mu J^\mu \equiv 0\), cf. homework #1.
3. Finally, let’s quantize the massive vector fields. Since classically the $-\mathbf{E}(\mathbf{x})$ fields are canonically conjugate momenta to the $\mathbf{A}(\mathbf{x})$ fields, the corresponding quantum fields $\hat{\mathbf{E}}(\mathbf{x})$ and $\hat{\mathbf{A}}(\mathbf{x})$ satisfy the canonical equal-time commutation relations

$$
\begin{align*}
[\hat{A}_i(x, t), \hat{A}_j(y, t)] &= 0, \\
[\hat{E}_i(x, t), \hat{E}_j(y, t)] &= 0, \\
[\hat{A}_i(x, t), \hat{E}_j(y, t)] &= -i\delta_{ij}\delta(3) (\mathbf{x} - \mathbf{y})
\end{align*}
$$

(in the $\hbar = c = 1$ units). The currents also become quantum fields $\hat{J}^\mu(\mathbf{x}, t)$, but they are composed of some kind of charged degrees of freedom rather than the vector fields in question. Consequently, the $\hat{J}^\mu(\mathbf{x}, t)$ commute with both $\hat{\mathbf{E}}(\mathbf{x})$ and $\hat{\mathbf{A}}(\mathbf{x})$ fields.

The classical $\mathbf{A}^0(\mathbf{x}, t)$ field does not have a canonical conjugate and its equation of motion does not involve time derivatives. In the quantum theory, $\hat{\mathbf{A}}^0(\mathbf{x}, t)$ satisfies a similar time-independent constraint

$$
m^2 \hat{\mathbf{A}}^0(\mathbf{x}, t) = \hat{\mathbf{J}}^0(\mathbf{x}, t) - \nabla \cdot \hat{\mathbf{E}}(\mathbf{x}, t).
$$

From the Hilbert space point of view, this is an operatorial identity rather than an equation of motion. Consequently, the commutation relations of the $\hat{\mathbf{A}}^0(\mathbf{x}, t)$ field follow from eqs. (9); in particular, $\hat{\mathbf{A}}^0(\mathbf{x}, t)$ commutes with the $\hat{\mathbf{E}}(\mathbf{x}, t)$ but does not commute with the $\hat{\mathbf{A}}(\mathbf{x}, t)$.

Finally, the Hamiltonian operator follows from the classical eq. (5), namely

$$
\hat{H} = \int d^3x \left\{ \frac{1}{2} \hat{\mathbf{E}}^2 + \hat{\mathbf{A}}_0 \left( \mathbf{J}_0 - \nabla \cdot \hat{\mathbf{E}} \right) - \frac{1}{2} m^2 \hat{\mathbf{A}}^2_0 + \frac{1}{2} \left( \nabla \times \hat{\mathbf{A}} \right)^2 + \frac{1}{2} m^2 \hat{\mathbf{A}}^2 - \mathbf{J} \cdot \hat{\mathbf{A}} \right\} \\
= \int d^3x \left\{ \frac{1}{2} \hat{\mathbf{E}}^2 + \frac{1}{2m^2} \left( \mathbf{J}_0 - \nabla \cdot \hat{\mathbf{E}} \right)^2 + \frac{1}{2} \left( \nabla \times \hat{\mathbf{A}} \right)^2 + \frac{1}{2} m^2 \hat{\mathbf{A}}^2 - \mathbf{J} \cdot \hat{\mathbf{A}} \right\}
$$

where the second line follows from the first and eq. (10).

Your task is to calculate the commutators $[\hat{A}_i(\mathbf{x}, t), \hat{H}]$ and $[\hat{E}_i(\mathbf{x}, t), \hat{H}]$ and write down the Heisenberg equations for the quantum vector fields. Make sure those equations are similar to the Hamilton equations for the classical fields.