1. First, a simple exercise about the Yukawa theory. For \( M_s > 2m_f \) the scalar particle becomes unstable: it decays into a fermion and an antifermion, \( S \rightarrow f + \bar{f} \).

(a) Calculate the tree-level decay rate \( \Gamma(S \rightarrow f + \bar{f}) \).

(b) In class, we have calculated \( \Sigma^{1\text{loop}}_{\Phi}(p^2) \). Show that for \( p^2 > 4m_f^2 \) this function has an imaginary part and calculate it for \( p^2 = M_s^2 + i\epsilon \).

Note: at this level, you may neglect the difference between \( m_f^{\text{bare}} \) and \( m_f^{\text{physical}} \).

(c) Verify that

\[
\text{Im} \Sigma^{1\text{loop}}_{\Phi}(p^2 = M_s^2 + i\epsilon) = -M_s \frac{\Gamma_{\text{tree}}(S \rightarrow f + \bar{f})}{2} \tag{1}
\]

and explain this relation in terms of the optical theorem.

The rest of this homework is about the scalar \( \lambda \phi^4 \) theory. As discussed in class, in this theory field strength renormalization begins at two-loop level. Specifically, the 1PI diagram

![1PI Diagram](image)

provides the leading contribution to the \( d\Sigma(p^2)/dp^2 \) and hence to the \( Z - 1 \). Your task is to evaluate this contribution. This is a difficult calculation, so proceed very carefully.

2. First, use Feynman parameters to write the product of 3 propagators as

\[
\prod_{j=1}^{3} \frac{i}{q_j^2 - m^2 + i0} = \int \int \int dx \, dy \, dz \, \delta(x + y + z - 1) \frac{2i^3}{(\mathcal{D})^3} \tag{3}
\]

where

\[
\mathcal{D} = xq_1^2 + yq_2^2 + zq_3^2 - m^2 + i0. \tag{4}
\]

Then impose \( q_3 \equiv p - q_1 - q_2 \) and shift the remaining 2 momentum variables from \( q_1 \) and
Let $q_2$ to $k_1 = q_1 + \cdots$ and $k_2 = q_2 + \cdots$ such that
\[
D = \alpha \times k_1^2 + \beta \times k_2^2 + \gamma \times p^2 - m^2 + i0
\] (5)
for some $(x, y, z)$–dependent coefficients $\alpha, \beta, \gamma$, for example
\[
\alpha = (x + z), \quad \beta = \frac{xy + xz + yz}{x + z}, \quad \gamma = \frac{xyz}{xy + xz + yz}.
\] (6)

Make sure the momentum shift has unit Jacobian $\partial(q_1, q_2)/\partial(k_1, k_2) = 1$. Warning: Do not set $p^2 = m^2$ at this stage.

3. Express the derivative $d\Sigma(p^2)/dp^2$ in terms of
\[
\int \int d^4k_1 d^4k_2 \frac{1}{D^4}.
\] (7)
Note that although this momentum integral diverges as $k_{1,2} \to \infty$, the divergence is logarithmic rather than quadratic.

4. To evaluate the momentum integral (7), first rotate both momenta $k_1$ and $k_2$ from Minkowski to Euclidean space, and then use dimensional regularization. You should get a formula looking like
\[
\frac{d\Sigma}{dp^2} = \int \int \int dxdydz \delta(x + y + z - 1) F(x, y, z) \times \\
\times \left\{ \frac{1}{\epsilon} + \log \frac{\mu^2}{m^2} + \text{const} + \log G(x, y, z; p^2/m^2) \right\}
\] (8)
for some rational functions $F$ and $G$ of the Feynman parameters (and in case of $G$, also of $p^2/m^2$). Here are some useful formulæ for this problem:
\[
\frac{6}{A^4} = \int_0^\infty dt \ t^3 e^{-At},
\] (9)
\[
\int \frac{d^Dk}{(2\pi)^D} e^{-ctk^2} = (4\pi ct)^{-D/2},
\] (10)
\[
\Gamma(2\epsilon) X^\epsilon = \frac{1}{2\epsilon} - \gamma_E + \frac{1}{2} \log X + O(\epsilon).
\] (11)

5. Before you evaluate the Feynman parameter integral (8)— which looks like a frightful mess — make sure it does not introduce its own divergences. That is, without actually
calculating the integrals

\[
\iiint dx dy dz \, \delta(x + y + z - 1) \, F(x, y, z)
\]  
(12)

and

\[
\iiint dx dy dz \, \delta(x + y + z - 1) \, F(x, y, z) \times \log G(x, y, z; p^2/m^2)
\]  
(13)

make sure that they converge. Pay attentions to the boundaries of the parameter space and especially to the corners where \(x, y \to 0\) while \(z \to 1\) (or \(x, z \to 0\), or \(y, z \to 0\)).

This calculation shows that

\[
\frac{d\Sigma}{dp^2} = \text{constant} \epsilon + \text{a finite function}(p^2) 
\]  
(14)

and hence

\[
\Sigma(p^2) = (\text{a divergent constant}) + (\text{another divergent constant}) \times p^2 + \text{a finite function}(p^2) 
\]  
(15)

up to the two-loop order. In fact, this behavior persists to all loops, so all the divergences of \(\Sigma(p^2)\) may be canceled with just two counterterms, \(\delta m\) and \(\delta Z \times p^2\).

6. Finally, let’s use bare perturbation theory (bare \(\lambda\) and bare \(m^2\) instead of the counterterms) and calculate field strength renormalization factor

\[
Z = \left[1 - \frac{d\Sigma}{dp^2}\right]^{-1} 
\]  
(16)

The derivative here should be evaluated at \(p^2 = M_{ph}^2\) — the physical mass\(^2\) of the scalar particle, but to the leading approximation we may let \(M_{ph}^2 \approx m^2\) and set \(p^2 = m^2\) in eq. (8). This should simplify the \(G(x, y, z)\) function, but the integral is still a big mess.

Do not try to evaluate the integrals (12) and (13) by hand — it would take way too much time. Instead, use Mathematica or equivalent software. To help it along, replace
the \((x, y, z)\) variables with \((w, \xi)\) according to

\[
x = x_i \times w, \quad y = (1 - \xi) \times w, \quad z = 1 - w,
\]

\[
\iiint_0^1 dx dy dz \, \delta(x + y + z - 1) = \int_0^1 dw \int_0^1 d\xi,
\]

then integrate over the \(w\) variable first and over the \(\xi\) second. Here is a couple of integrals I did this way you might find useful:

\[
\iiint dxdydz \, \delta(x + y + z - 1) \times \frac{xyz}{(xy + xz + yz)^3} = \frac{1}{2},
\]

\[
\iiint dxdydz \, \delta(x + y + z - 1) \times \frac{xyz}{(xy + xz + yz)^3} \times \log \frac{(xy + xz + yz)^3}{(xy + xz + yz - xyz)^2} = -\frac{3}{4}.
\]

Alternatively, you may evaluate the integrals like this numerically. In this case, don’t bother changing variables, just use a simple 2D grid spanning a triangle defined by \(x + y + z = 1, \ x, y, z \geq 0\); modern computers can sum up to \(10^8\) grid points in just a few seconds. But watch out for singularities at the corners of the triangle.