1. Consider the Wess–Zumino model, a QFT comprising a Majorana spinor $\Psi(x)$, a real scalar $\Phi_1(x)$, and a real pseudoscalar $\Phi_2(x)$, all massless. The Lagrangian is

$$\mathcal{L} = i\bar{\Psi} \gamma^\mu \partial_\mu \Psi + \frac{1}{2} (\partial_\mu \Phi_1)^2 + \frac{1}{2} (\partial_\mu \Phi_2)^2 - g\bar{\Psi}(\Phi_1 + i\gamma^5 \Phi_2)\Psi - \frac{\lambda}{8} (\Phi_1^2 + \Phi_2^2)^2, \quad (1)$$

and there is a global $U(1)$ chiral symmetry which acts as

$$\Psi \to \exp(i\theta\gamma^5)\Psi, \quad (\Phi_1 + i\Phi_2) \to \exp(-2i\theta) \times (\Phi_1 + i\Phi_2). \quad (2)$$

Wess and Zumino found that for $\lambda = g^2$, the renormalization of this theory is particularly simple, but at first they did not know why. Salam and Strathdee realized there must be a symmetry behind this simplicity, and after working very hard to find it, they discovered the supersymmetry.

Thanks to the chiral symmetry, the WZ model needs only 5 independent counterterms, namely $\delta^g$, $\delta^\lambda$, $\delta^Z_{\Phi_1}$, $\delta^Z_{\Phi_2}$, and $\delta^m_{\psi}$, but no $\delta^m_{\Psi}$! In general, $\delta^m_{\Phi} = O(\Lambda^2)$ while the other counterterms are $O(\log(\Lambda/E))$.

(a) For $\lambda = g^2$, the quadratic divergence of the two-scalar 1PI amplitude vanishes. Instead, $\Sigma_{\Phi}(p^2) = p^2 \times O(\log \Lambda^2/p^2)$ and hence $\delta^m_{\Phi} = 0$ while $\delta^Z_{\Phi} = O(\log \Lambda/E)$. Show that this is true at the one-loop level.

Note: Feynman rules for the Majorana fermions are similar to those for the Dirac fermions (same propagators, vertices, and external leg factors), but there is an extra factor $\frac{1}{2}$ for each closed fermionic loop. (i.e., $-\frac{1}{2} \text{tr}(\cdots)$ instead of $-\text{tr}(\cdots)$).

(b) Next, calculate the infinite parts of the other 4 counterterms at the one-loop level. Proceed similarly to homework #17, and do not hesitate to recycle similar calculations instead of redoing them from scratch. Do not assume $\lambda = g^2$ at this stage.

(c) Calculate the anomalous dimensions of the scalar and fermionic fields to order $O(g^2, \lambda)$ and show that $\gamma_{\phi} = \gamma_{\psi}$. Note: at the one-loop level this is true for any $\lambda$, but at the higher loop levels $\gamma_{\phi} = \gamma_{\psi}$ only when $\lambda = g^2$. 

(d) Calculate the beta-functions $\beta_g(g, \lambda)$ and $\beta_\lambda(g, \lambda)$ to one-loop order for general $\lambda$ and $g$. Then show that

$$ \text{for } \lambda = g^2, \quad \beta_\lambda(\lambda = g^2) = 2g \times \beta_g(\lambda = g^2). \quad (3) $$

Note: because of supersymmetry, this relation holds true to all orders of the perturbation theory. But in this exercise, you should check it at the one-loop level only.

(e) Show that the relation (3) implies that if $\lambda(E_0) = g^2(E_0)$ for any particular energy $E_0$, then $\lambda(E) = g^2(E)$ for all energies $E$.

(f) Finally, consider the renormalization group flow in the $(g^2, \lambda)$ plane. In the $\text{UV} \rightarrow \text{IR}$ direction, is the $\lambda = g^2$ line attractive or repulsive?

2. And now a reading assignment: *Quantum Mechanics and Path Integrals* by Feynman & Hibbs. Read all you can about care and use of Path Integrals. After the break, I will talk about “path” integrals in QFT, and it would help if you already know something about path integrals in the ordinary QM.