1. Let’s start with the pion decay. In class I have explained how the QED anomaly of the axial quark current makes the neutral pion decay into two photons, \( \pi^0 \rightarrow \gamma \gamma \).

(a) Finish the calculation of the neutral pion’s lifetime. For your information, pion decay constant is \( f_\pi \approx 93 \text{ MeV} \) and the mass of \( \pi^0 \) is about 135 MeV.

The \( f_\pi \) is called the pion decay constant because it controls the decay of the charged pions into muons and neutrinos, \( \pi^+ \rightarrow \mu^+ \nu_\mu \) and \( \pi^- \rightarrow \mu^- \bar{\nu}_\mu \). This decay is due to weak interaction, and since \( M_\pi \ll M_W \) we may use Fermi’s current-current interactions

\[
\mathcal{L} = -2\sqrt{2}G_F J^+_L J^-_L \quad \text{where} \quad J^+_L = \frac{1}{2}(J^+_V - J^+_A).
\]

(1)

For the pion decay process, one of the currents annihilates the charged pion while the other creates the lepton pair.

(b) Show that the decay amplitude is

\[
\mathcal{M}(\pi^+ \rightarrow \mu^+ \nu_\mu) = G_f f_\pi p^0(\pi) \times \bar{\nu}(\mu)(1 - \gamma^5)\gamma_\alpha u(\nu).
\]

(2)

(c) Sum over the fermion spins and calculate the decay rate \( \Gamma(\pi^+ \rightarrow \mu^+ \nu_\mu) \). Show that this decay rate would vanish if both leptons are massless, and that’s why the pion decays mostly into \( \mu^+ \nu_\mu \) rather than \( e^+ \nu_e \). Specifically,

\[
\frac{\Gamma(\pi^+ \rightarrow e^+ \nu_e)}{\Gamma(\pi^+ \rightarrow \mu^+ \nu_\mu)} = \frac{M_e^2 (1 - (M_e/M_\pi)^2)^2}{M_\mu^2 (1 - (M_\mu/M_\pi)^2)^2} \approx 1.2 \cdot 10^{-4}.
\]

(3)

(d) Explain this preference for the heavier lepton in terms of mismatch between chirality and helicity.
2. Now consider the axial anomaly in a non-abelian gauge theory, for example QCD with one massless quark flavor. In this case,

\[ \partial_\mu \left( J_5^\mu = \bar{\Psi} \gamma^5 \gamma^\mu \Psi \right) = \frac{g^2}{16\pi^2} \epsilon^{\alpha\beta\gamma\delta} \text{tr} \left( F_{\alpha\beta} F_{\gamma\delta} \right) \]  

(4)

where \( F_{\mu\nu} \) is the non-abelian gauge field strength.

(a) Expand the right hand side of eq. (4) into 2–gluon, 3–gluon, and 4–gluon terms and show that the 4–gluon term vanishes identically.

Hint: Use the cyclic symmetry of the trace.

The two-gluon anomaly term obtains from the triangle diagrams

\[ \text{triangle diagrams} + \text{gluon permutation.} \]  

(5)

This works exactly as discussed in class for the QED, except in QCD we should trace \( F_{\alpha\beta} F_{\gamma\delta} \) over the quark colors. But in QCD there is also the three-gluon anomaly (cf. part (a)) which obtains from the quadrangle diagrams

\[ \text{quadrangle diagrams} + \text{gluon permutations.} \]  

(6)

(b) Evaluate the quadrangle diagrams using the Pauli–Villars regularization and derive the three-gluon anomaly in QCD.
3. In any even spacetime dimension \( d = 2n \), a massless Dirac fermion has an axial symmetry 
\[ \Psi(x) \rightarrow \exp(i\theta \Gamma)\Psi(x) \] 
where \( \Gamma \) generalizes the \( \gamma^5 \). Classically, the axial current \( J_A^\mu = \bar{\Psi} \Gamma \gamma^\mu \Psi \) is conserved, but when the fermion is coupled to a gauge field — abelian or non-abelian — the axial symmetry is broken by the anomaly and

\[
\partial_\mu J^\mu_A = -\frac{2}{n!} \left( \frac{g}{4\pi} \right)^n \epsilon^{\alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_n \beta_n} \text{tr} \left( F_{\alpha_1 \beta_1} F_{\alpha_2 \beta_2} \cdots F_{\alpha_n \beta_n} \right). \tag{7}
\]

In class, we have derived this formula for \( d = 4 \) by formally calculating the Jacobian 
\( \text{Det}(2i\theta \gamma^5) \) of the fermionic path integral for the axial symmetry. In this exercise, you should similarly derive the anomaly equation (7) for any even spacetime dimension \( d = 2n \).

For your information, in \( 2n \) Euclidean dimensions \( \{ \gamma^\mu, \gamma^\nu \} = +2\delta^{\mu\nu}, \Gamma = i^{n-2}\gamma^1\gamma^2 \cdots \gamma^{2n}, \) \( \{ \Gamma, \gamma^\mu \} = 0, \Gamma^2 = +1, \) and \( \text{tr} (\Gamma^{\alpha_1} \gamma^{\beta_1} \cdots \gamma^{\omega}) = 2^n i^{2-n} \epsilon^{\alpha_1 \beta_1 \cdots \omega} \) (for \( 2n = d \) matrices \( \gamma^\alpha \cdots \gamma^\omega \)).