Problem 1(a): 
As discussed in class for the massless case (EM), \( \partial \mathcal{L} / \partial (\partial_\mu A_\nu) = -F^{\mu\nu} \).

Clearly, \( \partial \mathcal{L} / \partial (A_\nu) = +m^2 A^\nu - J^\nu \). Hence, the Euler–Lagrange field equation is

\[
-\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} + \frac{\partial \mathcal{L}}{\partial (A_\nu)} \equiv \partial_\mu F^{\mu\nu} + m^2 A^\nu - J^\nu = 0, \tag{S.1}
\]

or in terms of \( A^\nu \) and their explicit derivatives,

\[
\partial^2 A^\nu - \partial^\nu (\partial_\mu A^\mu) + m^2 A^\nu - J^\nu = 0. \tag{S.2}
\]

Problem 1(b):
Take the divergence \( \partial_\nu \) of the field equation (S.2); the first two terms cancel out while the rest becomes

\[
m^2 \partial_\nu A^\nu - \partial_\nu J^\nu = 0. \tag{S.3}
\]

In the massless case, this equation enforces the current conservation \( \partial_\nu J^\nu = 0 \) regardless of the 4–vector potential \( A^\nu(x) \), but there is no such constraint in the massive case at hand. Instead, eq. (S.3) simply relates the current divergence to the 4–potential divergence. In particular, if the current happens to satisfy \( \partial_\nu J^\nu \), then — and only then — eq. (S.3) requires \( \partial_\nu A^\nu = 0 \) as well. Consequently, the field equation (S.2) simplifies to \( (\partial^2 + m^2)A^\nu = J^\nu \). Q.E.D.
Problem 2(a):
The definition (3) is manifestly symmetric with respect to cyclic permutations of indices $\lambda, \mu, \nu$, thus

$$H_{\lambda\mu\nu} = H_{\mu\nu\lambda} = H_{\nu\lambda\mu}. \quad (S.4)$$

Hence, to prove the total antisymmetry of the $H_{\lambda\mu\nu}$ tensor it is enough to show that $H_{\lambda\nu\mu} = -H_{\lambda\mu\nu}$ — antisymmetry with respect to other index pairs then follows by the cyclic symmetry (S.4). And indeed, antisymmetry of the $B$ tensor leads to

$$H_{\lambda\nu\mu} = \partial_\lambda B_{\nu\mu} + \partial_\nu B_{\mu\lambda} + \partial_\mu B_{\lambda\nu} = -\partial_\lambda B_{\mu\nu} - \partial_\nu B_{\lambda\nu} - \partial_\mu B_{\nu\lambda} \quad (S.5)$$

and hence total antisymmetry of the $H$ tensor. Q.E.D.

Alternative proof:
Let us redefine the $H$ tensor as

$$H_{\lambda\mu\nu} = \frac{1}{2}\partial_{[\lambda} B_{\mu\nu]} \quad (S.6)$$

where $[\lambda\mu\nu]$ imply total antisymmetrization with respect to the $\lambda, \mu, \nu$, i.e. summing over all possible permutations of those indices with appropriate signs. Obviously, this new definition makes $H_{\lambda\mu\nu}$ a totally antisymmetric tensor.

To see that the new definition (S.6) is equivalent to the old definition (3) we use the fact that $B_{\mu\nu}$ is itself antisymmetric. Consequently, there is no need to antisymmetrize $\partial_{[\lambda} B_{\mu\nu]}$ with respect to the two indices belonging to the $B$ tensor, thus

$$H_{\lambda\mu\nu} = \frac{1}{2}(\partial_\lambda B_{\mu\nu} - \partial_\lambda B_{\nu\mu}) + \frac{1}{2}(\partial_\mu B_{\nu\lambda} - \partial_\mu B_{\lambda\nu}) + \frac{1}{2}(\partial_\nu B_{\lambda\mu} - \partial_\nu B_{\mu\lambda}) = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}, \quad (S.7)$$

exactly as in eq. (3).
Problem 2(b):
Thanks to eq. (S.6),
\[ \partial_{[\kappa}H_{\lambda\mu\nu]} = \frac{1}{2}\partial_{[\kappa}\partial_{\lambda}B_{\mu\nu]} , \quad (S.8) \]
which vanishes because spacetime derivatives \( \partial_{\kappa} \) commute with each other and hence \( \partial_{[\kappa}\partial_{\lambda]} = 0 \). Thus eq. (3) — or equivalently (S.6) — leads to Jacobi identity
\[ \partial_{[\kappa}H_{\lambda\mu\nu]} = 0. \quad (S.9) \]

Note that \( \partial_{[\kappa}H_{\lambda\mu\nu]} \) stand for signed sum of \( 4! = 24 \) terms according to permutations of the indices \( \kappa, \lambda, \mu, \nu \). Fortunately, total antisymmetry of the \( H \) tensor means that there is a 6-fold redundancy and only 4 of those 24 terms are different, thus
\[ \frac{1}{6} \partial_{[\kappa}H_{\lambda\mu\nu]} = \partial_{\kappa}H_{\lambda\mu\nu} - \partial_{\lambda}H_{\mu\nu\kappa} + \partial_{\mu}H_{\nu\kappa\lambda} - \partial_{\nu}H_{\kappa\lambda\mu} . \quad (S.10) \]

Consequently, the Jacobi identity (S.9) for the \( H_{\lambda\mu\nu}(x) \) field may be written as (4).

Problem 2(c):
Given the Lagrangian (5) as a function of \( B_{\mu\nu} \) fields and their derivatives, we have
\[ \frac{\partial L(B, \partial B)}{\partial B_{\mu\nu}} = 0 \quad (S.11) \]
while
\[ \frac{\partial L(B, \partial B)}{\partial (\partial_{\lambda}B_{\mu\nu})} = \frac{1}{6} H^{\alpha\beta\gamma} \times \frac{\partial H_{\alpha\beta\gamma}}{\partial (\partial_{\lambda}B_{\mu\nu})} = \frac{1}{6} H^{\alpha\beta\gamma} \times \frac{1}{2}\delta^{[\lambda}_{\alpha} \delta_{\beta}^{\mu} \delta_{\gamma}^{\nu]} = \frac{1}{2} H_{\lambda\mu\nu} . \quad (S.12) \]

Consequently, the Euler–Lagrange field equations
\[ \partial_{\lambda} \left( \frac{\partial L(B, \partial B)}{\partial (\partial_{\lambda}B_{\mu\nu})} \right) - \frac{\partial L(B, \partial B)}{\partial B_{\mu\nu}} = 0 \quad (S.13) \]
for the \( B \) fields become
\[ \partial_{\lambda}H^{\lambda\mu\nu} = 0. \quad (S.14) \]
Problem 2(d):
Rewriting eq. (6) as
\[ B_{\mu\nu}'(x) = B_{\mu\nu}(x) + \partial_{[\mu} \Lambda_{\nu]}(x), \]  
we have
\[ H_{\lambda\mu\nu}'(x) = \frac{1}{2} \partial_{[\lambda} B_{\mu\nu]}(x) \]
\[ = \frac{1}{2} \partial_{[\lambda} B_{\mu\nu]}(x) + \frac{1}{2} \partial_{\lambda} \partial_{\mu} \Lambda_{\nu]}(x) \]
\[ = H_{\lambda\mu\nu}(x) + 0, \]
where the last equality follows from \( \partial_{[\lambda} \partial_{\mu]} = 0 \). Thus, the tension fields \( H_{\lambda\mu\nu}(x) \) — and hence the Lagrangian (5) — are invariant under the gauge transforms (6).

Problem 2(e):
Proceeding similarly to part (b), we have
\[ \partial_{[\lambda} G_{\mu_1\mu_2\cdots\mu_{p+1]}(x) = \frac{1}{p!} \partial_{[\lambda} \partial_{\mu_1} C_{\mu_2\cdots\mu_{p+1]}(x) = 0 \]  
(S.17)
because \( \partial_{[\lambda} \partial_{\mu_1} = 0 \). This regardless of any equations obeyed or not obeyed by the \( C(x) \) potentials, their very existence implies the Jacobi identity
\[ \partial_{[\lambda} G_{\mu_1\mu_2\cdots\mu_{p+1]}(x) = 0 \]  
(S.18)
for the tension fields \( G(x) \).

As to the equations of motion, the Lagrangian (8) has derivatives
\[ \frac{\partial L(C, \partial C)}{\partial C_{\mu_1\cdots\mu_p}} = 0, \]
\[ \frac{\partial L(C, \partial C)}{\partial (\partial_{[\lambda} C_{\mu_1\cdots\mu_p]/}} = \frac{(-1)^p}{(p+1)!} G^{\alpha_1\cdots\alpha_{p+1}} \times \frac{\partial G_{\alpha_1\cdots\alpha_{p+1}}}{\partial (\partial_{[\lambda} C_{\mu_1\cdots\mu_p])} \]
\[ = \frac{(-1)^p}{(p+1)!} G^{\alpha_1\cdots\alpha_{p+1}} \times \frac{1}{p!} \delta_{\alpha_1} \delta_{\alpha_2} \delta_{\alpha_3} \cdots \delta_{\alpha_{p+1}} \]
\[ = \frac{(1)^p}{p!} G^{\lambda\mu_1\cdots\mu_p}. \]  
(S.19)
Hence, the Euler–Lagrange field equations

\[ \partial_{\lambda} \left( \frac{\partial L(C, \partial C)}{\partial \partial_{\lambda} C_{\mu_1 \cdots \mu_p}} \right) - \frac{\partial L(C, \partial C)}{\partial C_{\mu_1 \cdots \mu_p}} = 0 \quad (S.20) \]

for the \( C_{\mu_1 \cdots \mu_p}(x) \) fields become (up to an overall coefficient)

\[ \partial_{\lambda} G^{\lambda \mu_1 \cdots \mu_p}(x) = 0. \quad (S.21) \]

**Problem 2(f):**
Under gauge transformations, the \( C \) tensor potential changes by

\[ \Delta C_{\mu_1 \cdots \mu_p}(x) = \frac{1}{(p-1)!} \partial_{[\mu_1} \Lambda_{\mu_2 \cdots \mu_p]}(x) \quad (S.22) \]

for some arbitrary \((p-1)\)-index antisymmetric tensor \( \Lambda_{\mu_2 \cdots \mu_p}(x) \). Hence the \( G \) tension tensor changes by

\[ \Delta G_{\mu_1 \mu_2 \cdots \mu_p+1}(x) = \frac{1}{p!} \partial_{[\mu_1} \Delta C_{\mu_2 \cdots \mu_p+1]}(x) = \frac{1}{(p-1)!} \partial_{[\mu_1} \partial_{\mu_2} \Lambda_{\mu_3 \cdots \mu_p+1]}(x), \quad (S.23) \]

which vanishes because \( \partial_{[\mu_1} \partial_{\mu_2]} = 0 \). Thus, the tension tensor \( G \) is gauge invariant, and hence the Lagrangian (8) is also gauge invariant. \( \text{Q.E.D.} \)

**Mathematical Supplement to Problem 2: Differential Forms.**
Mathematics of various antisymmetric tensor fields becomes much simpler in the language of differential forms. Students interested in string theory should master this language and then go ahead and learn as much differential geometry and topology as they can; take a class on the subject or at least read a book. A quick introduction to differential forms is available at WikiSlice at


and related web pages.
Consider a space or spacetime of dimension \( D \); it can be Euclidean or Minkowski, flat or curved. A differential form of rank \( p \leq D \) combines a tensor with \( p \) indices and a differential suitable for integration over a manifold of dimension \( p \) (a line for \( p = 1 \), a surface for \( p = 2 \), etc., etc.). For example,

\[
A = A_\mu(x) \, dx^\mu, \quad B = B_{\mu\nu}(x) \, dx^\mu \, dx^\nu, \quad C = C_{\lambda\mu\nu}(x) \, dx^\lambda \, dx^\mu \, dx^\nu, \ldots \tag{S.24}
\]

For \( p = 2 \) a 2-form should be integrated over an oriented surface, so the order of \( dx^\mu \) and \( dx^\nu \) matters; in fact they anticommute so \( dx^\mu \, dx^\nu = -dx^\nu \, dx^\mu \). Likewise, the volume differential \( dx^\lambda \, dx^\mu \, dx^\nu \) is totally antisymmetric with respect to permutation of indices \( \lambda\mu\nu \). Consequently, in eq. (S.24), the \( B_{\mu\nu}(x) \) tensor is antisymmetric and the \( C_{\lambda\mu\nu}(x) \) tensor is totally antisymmetric in all 3 indices. And a general form of rank \( p \)

\[
E = E_{\mu_1\mu_2\cdots\mu_p}(x) \, dx^{\mu_1} \, dx^{\mu_2} \cdots dx^{\mu_p} \tag{S.25}
\]

involves a \( p \)-index totally antisymmetric tensor \( E_{\mu_1\mu_2\cdots\mu_p}(x) \).

The exterior derivative of a rank-\( p \) form \( E \) is a form \( dE \) of rank \( p + 1 \) defined as

\[
dE = \partial_\lambda E_{\mu_1\mu_2\cdots\mu_p}(x) \, dx^\lambda \, dx^{\mu_1} \, dx^{\mu_2} \cdots dx^{\mu_p}, \tag{S.26}
\]

but this compact formula hides the antisymmetrization due to anticommutativity of the \( dx^\mu \) differentials. In the antisymmetric tensor form, \( J = dE \) means

\[
J_{\mu_1\cdots\mu_{p+1}}(x) = \frac{1}{p!} \partial_{[\mu_1} E_{\mu_2\cdots\mu_{p+1}]}(x) = \sum_{j=1}^{p+1} (-1)^{j-1} \partial_{\mu_j} E_{\mu_1\cdots\mu_j\cdots\mu_{p+1}}(x). \tag{S.27}
\]

The exterior derivative generalizes the 3D notions of gradient, curl, and divergence. Indeed, a scalar \( \phi(x) \) is a 0-form and its gradient \( \nabla \phi \) is a vector defining a 1-form \( (\nabla \phi)_\mu \, dx^\mu = d\phi \). Likewise, for a vector \( \vec{A}(x) \) and its curl \( \vec{B}(x) = \nabla \times \vec{A}(x) \) we have a 1-form \( A = A_\mu(x) \, dx^\mu \) and a 2-form \( B = B_{\mu\nu}(x) \, dx^\mu \, dx^\nu = dA \) where \( B_{\mu\nu} = \epsilon_{\mu\nu\lambda} B^\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu \). (Note that for \( D = 3 \) an antisymmetric tensor is equivalent to a vector.) Finally, for a vector \( \vec{E}(x) \) and its divergence \( f(x) = \nabla \cdot \vec{E} \) we have an exterior derivative relation \( f = dE \) where \( E = E^\lambda(x) \epsilon_{\lambda\mu\nu} dx^\mu dx^\nu \) is a 2-form and \( f = f(x) \epsilon_{\lambda\mu\nu} dx^\lambda dx^\mu dx^\nu \) is a 3-form.
The most important property of the exterior derivative is its nilpotency: for any differential form \( E \), \( ddE = 0 \). This rule generalizes \( \nabla \times (\nabla \phi) = 0 \) and \( \nabla \cdot (\nabla \times \vec{A}) = 0 \). The proof is very simple: If \( E \) is a form of rank \( p \), \( J = dE \) is a form of rank \( p + 1 \), and \( K = dJ \) is a form of rank \( p + 2 \), then applying eq. (S.27) twice, we have

\[
K_{\lambda\mu\nu_1\cdots\nu_p}(x) = \frac{1}{(p+1)!} \partial_{[\lambda} J_{\mu\nu_1\cdots\nu_p]}(x)
= \frac{1}{(p+1)!p!} \partial_{[\lambda} \partial_{[\mu} E_{\nu_1\cdots\nu_p]}(x)
= \frac{1}{p!} \partial_{[\lambda} \partial_{\mu} E_{\nu_1\cdots\nu_p]}(x)
= 0
\]

where the last equality follow from \( \partial_{[\lambda} \partial_{\mu]} = 0 \).

The application of the differential form language to electromagnetic fields and to the antisymmetric tensor fields in this homework is completely straightforward. In electromagnetism we work in Minkowski spacetime and identify the 4–vector potential \( A_\mu(x) \) with a 1–form \( A = A_\mu(x) dx^\mu \) and the tension tensor \( F_{\mu\nu}(x) \) with a 2–form \( F = F_{\mu\nu}(x) dx^\mu dx^\nu \). Clearly, \( F \) is the exterior derivative of \( A \):

\[
F = dA \iff F_{\mu\nu} = \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

The Jacobi identity \( \partial_{[\lambda} F_{\mu\nu\rho]} = 0 \) is simply \( dF = 0 \), which follows from \( F = dA \) and \( dF = ddA = 0 \) by nilpotency of \( d \). The gauge transform of the potentials is \( A' = A + d\Lambda \) (where \( \Lambda \) is a 0–form, \( i.e. \) a scalar field), and the gauge invariance of the tension fields is simply

\[
F' = dA' = dA + dd\Lambda = F + 0
\]

because \( dd\Lambda = 0 \).

Similarly, for the tensor potential \( B_{\mu\nu}(x) \) and the tension tensor \( H_{\lambda\mu\nu} \) in parts (a) through (d) of problem 2, we have a 2–form \( B = B_{\mu\nu}(x) dx^\mu dx^\nu \) and a 3–form \( H = H_{\lambda\mu\nu}(x) dx^\lambda dx^\mu dx^\nu \). Clearly, eq. (3) for the tension tensor translates to \( H = dB \), which immediately gives rise to the Jacobi identity \( dH = 0 \) because \( dB = 0 \). Translating back
to the tensor language, \( dH = 0 \) means eq. (4). And the gauge transform (6) is simply
\[
B' = B + d\Lambda^{(1)}
\]
where \( \Lambda^{(1)} \) is an arbitrary 1–form; the tension form \( H \) is gauge invariant because \( dd\Lambda^{(1)} = 0 \).

Finally, in parts (e) and (f) of problem 2, the totally-antisymmetric tensor potential
\[
C_{\mu_1 \cdots \mu_p}(x)
\]
with \( p \) indices corresponds to a form \( C \) of rank \( p \) and the tension tensor (7) corresponds to a form \( G = dC \) of rank \( p + 1 \). The Jacobi identity is \( dG = 0 \), which follows from \( ddC = 0 \). And the gauge transform (9) is
\[
C' = C + d\Lambda^{(p-1)}
\]
for an arbitrary rank \( p - 1 \) form \( \Lambda^{(p-1)} \); the tension form \( G \) is gauge invariant because \( dd\Lambda^{(p-1)} = 0 \).

The Lagrangians and the equations of motion for the EM and tensor fields may also be written in the differential form language, but this is less convenient, so I am not doing it here.
Problem 3(a):
Let \( \Delta T^{\mu\nu} = \partial_\lambda K^{[\lambda\mu]\nu} \). Regardless of the specific form of the \( K^{[\lambda\mu]\nu}(\phi, \partial \phi) \) tensor, its antisymmetry with respect to its first two indices implies

\[
\partial_\mu \Delta T^{\mu\nu} = \partial_\mu \partial_\lambda K^{[\lambda\mu]\nu} = 0 \tag{S.31}
\]

and hence the first eq. (12). Furthermore,

\[
\int d^{3}x \left( \Delta T^{0\nu} = \partial_i K^{i0\nu} \right) = \oint_{\text{boundary of space}} d^2 \text{Area}_i K^{i0\nu} \longrightarrow 0 \tag{S.32}
\]

when the integral is taken over the whole space, hence the second eq. (12).

Problem 3(b):
In the Noether’s formula (10) for the stress-energy tensor, \( \phi_\alpha \) stand for the independent fields, however labeled. In the electromagnetic case, the independent fields are components of the 4–vector \( A_\lambda(x) \), hence

\[
T^{\mu\nu}_{\text{Noether\,(EM)}} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L} = -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}. \tag{S.33}
\]

While the second term here is clearly both gauge invariant and symmetric in \( \mu \leftrightarrow \nu \), the first term is neither.

Problem 3(c):
Clearly, one can easily restore both symmetry and gauge invariance of the electromagnetic stress-energy tensor by replacing \( \partial^\nu A_\lambda \) in eq. (S.33) with \( F^\nu_\lambda \), hence eq. (14). The correction amounts to

\[
\Delta T^{\mu\nu} = T^{\mu\nu}_{\text{true}} - T^{\mu\nu}_{\text{Noether}} = -F^{\mu\lambda} \left( F^\nu_\lambda - \partial^\nu A_\lambda = -\partial_\lambda A^\nu \right) \tag{S.34}
\]

where the last term on the right hand side vanishes for the free electromagnetic field (which
satisfies $\partial_\alpha F^{\mu\lambda} = 0$). Consequently,

$$T^{\mu\nu} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda K^{\mu\nu} \quad \text{where} \quad K^{\mu\lambda\nu} = F^{\mu\lambda} A^\nu = -K^{\mu\lambda\nu}$$  \hspace{1cm} (S.35)

in perfect agreement with eq. (11).

**Problem 3(d):**

Let’s start with the Lagrangian (13). In component form,

$$F^{i0} = -F^{0i} = E^i, \quad F^{ij} = -\epsilon^{ijk} B^k.$$  \hspace{1cm} (S.36)

Therefore, $F^{i0} F_{i0} = F^{0i} F_{0i} = -E^i E^i$ where the minus sign comes from raising one space index. Likewise, $F^{ij} F_{ij} = +\epsilon^{ijk} B^k \epsilon^{ij\ell} B^\ell = +2B^k B^k$ where the plus sign comes from raising two space indices at once. Thus,

$$\mathcal{L} = -\frac{1}{4} \left( F^{\mu\nu} F_{\mu\nu} = F^{i0} F_{i0} + F^{0i} F_{0i} + F^{ij} F_{ij} \right) = \frac{1}{2} \left( E^2 - B^2 \right).$$  \hspace{1cm} (S.37)

Consequently, eq. (14) for the energy density gives

$$\mathcal{H} = T^{00} = -F^{0i} F_{i0} - \mathcal{L} = +E^2 - \frac{1}{2} \left( E^2 - B^2 \right) = \frac{1}{2} \left( E^2 + B^2 \right)$$  \hspace{1cm} (S.38)

in agreement with the standard electromagnetic formulæ (note the $c = 1$, rationalized units here). Likewise, the energy flux and the momentum density are

$$S^i = T^{i0} = T^{0i} = -F^{0j} F_{j}^i = -(-E^j)(+\epsilon^{ijk} B^k) = +\epsilon^{ijk} E^j B^k = (E \times B)^i,$$  \hspace{1cm} (S.39)

in agreement with the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ (again, in the $c = 1$, rationalized units).

Finally, the (3–dimensional) stress tensor is

$$T^{ij}_{\text{EM}} = -F^{i\lambda} F^ j _\lambda - g^{ij} \mathcal{L} = -F^{i0} F^j_0 - F^{ik} F^j_k + \delta^{ij} \mathcal{L}$$

$$= -E^i E^j + \epsilon^{ik\ell} B^\ell \epsilon^{jkm} B^m + \frac{1}{2} \delta^{ij} \left( E^2 - B^2 \right)$$

$$= -E^i E^j - B^i B^j + \frac{1}{2} \delta^{ij} \left( E^2 + B^2 \right).$$  \hspace{1cm} (S.40)
Problem 3(e):
In a sense, eq. (16) follows from eq. (S.34), but it is just as easy to derive it directly from Maxwell equations. Starting with eq. (14), we immediately have

$$
\partial_\mu T^{\mu\nu}_{\text{EM}} = - \left( \partial_\mu F^{\mu\lambda} \right) F_\lambda^{\nu} - F^{\mu\lambda} \left( \partial_\mu F_\lambda^{\nu} \right) + \frac{1}{2} F_{\kappa\lambda} \left( \partial^\nu F^{\kappa\lambda} \right). \tag{S.41}
$$

Using the antisymmetry $F^{\mu\lambda} = - F^{\lambda\mu}$, we rewrite the second term on the right hand side as

$$
- F^{\mu\lambda} \partial_\mu F_\lambda^{\nu} = + F_{\mu\lambda} \partial^{\mu} F^{\lambda\nu} = + F_{\mu\lambda} \partial^\lambda F^{\nu\mu} = \frac{1}{2} F_{\mu\lambda} \left( \partial^\lambda F^{\nu\mu} + \partial^{\mu} F^{\lambda\nu} \right) = - \frac{1}{2} F_{\mu\lambda} \left( \partial^\nu F^{\mu\lambda} \right) \tag{S.42}
$$

where the last equality follows from the homogeneous Maxwell equation

$$
\epsilon_{\kappa\lambda\mu\nu} \partial^\lambda F^{\mu\nu} = 0 \iff \partial^\lambda F^{\nu\mu} + \partial^{\mu} F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} = 0. \tag{S.43}
$$

Consequently, the second and the third terms on the right hand side of eq. (S.41) cancel each other and we are left with the first term only. Thus,

$$
\partial_\mu T^{\mu\nu}_{\text{EM}} = - \left( \partial_\mu F^{\mu\lambda} \right) F_\lambda^{\nu} = - J^\lambda F_\lambda^{\nu} \tag{S.44}
$$

where the second equality comes from the in-homogeneous Maxwell equation $\partial_\mu F^{\mu\lambda} = J^\lambda$. This proves eq. (16), and eq. (17) follows from that and eq. (15). \textbf{Q.E.D.}