Problem 1(a):
As discussed in class, Euler–Lagrange equations for charged fields can be written in a manifestly covariant form as

\[ D_\mu \frac{\partial L}{\partial (D_\mu \phi)} - \frac{\partial L}{\partial \phi} = 0. \]  

(S.1)

In particularly, for \( \phi = \Phi \), we have

\[ \frac{\partial L}{\partial (D_\mu \Phi)} = D^\mu \Phi^*, \quad \frac{\partial L}{\partial \Phi} = -m^2 \Phi^*, \]

which gives us

\[ D_\mu D^\mu \Phi^* + m^2 \Phi^* = 0. \]  

(S.2)

Likewise, for \( \phi = \Phi^* \) we have

\[ \frac{\partial L}{\partial (D_\mu \Phi^*)} = D^\mu \Phi, \quad \frac{\partial L}{\partial \Phi^*} = -m^2 \Phi, \]

and therefore

\[ D_\mu D^\mu \Phi + m^2 \Phi = 0. \]  

(S.3)

As for the vector fields \( A_\nu \), the Lagrangian (1) depends on \( \partial_\mu A_\nu \) only through \( F_{\mu \nu} \), which gives us the usual Maxwell equation

\[ \partial_\mu F^{\mu \nu} = J^\nu \quad \text{where} \quad J^\nu \equiv -\frac{\partial L}{\partial A_\nu}. \]  

(S.4)

To obtain the current \( J^\nu \), we notice that the covariant derivatives of the charged fields \( \Phi \) and
Φ* depend on the gauge field:

\[
\frac{\partial D_\mu \Phi}{\partial A_\nu} = iq\delta^\nu_\mu \Phi, \quad \frac{\partial D_\mu \Phi^*}{\partial A_\nu} = -iq\delta^\nu_\mu \Phi^*.
\] (S.5)

Consequently,

\[
J^\nu = -\frac{\partial L}{\partial D_\nu \Phi} \times (iq\Phi) - \frac{\partial L}{\partial D_\nu \Phi^*} \times (-iq\Phi^*)
\]

\[
= -iq (\Phi D^\nu \Phi^* - \Phi^* D^\nu \Phi).
\] (S.6)

Note that all derivatives on the last line here are gauge-covariant, which makes the current \( J^\nu \) gauge invariant. In a non-covariant form,

\[
J^\nu = iq\Phi^* D^\nu \Phi - iq\Phi D^\nu \Phi^* - 2q^2 \Phi^* \Phi A^\nu.
\] (S.7)

To prove the conservation of this current, we use the Leibniz rule for covariant derivatives, \( D_\nu (XY) = XD_\nu Y + YD_\nu X \). This gives us

\[
\partial_\mu (\Phi^* D^\mu \Phi) = \partial_\mu (\Phi^* D^\mu \Phi) = (D_\mu \Phi^*)(D^\mu \Phi) + \Phi^* (D_\mu D^\mu \Phi),
\]

\[
\partial_\mu (\Phi D^\mu \Phi^*) = \partial_\mu (\Phi D^\mu \Phi^*) = (D_\mu \Phi)(D^\mu \Phi^*) + \Phi (D_\mu D^\mu \Phi^*),
\] (S.8)

and hence in light of eq. (S.6) for the current,

\[
\partial_\nu J^\nu = -iq \left( (D_\nu \Phi)(D^\nu \Phi^*) + \Phi (D_\nu D^\nu \Phi^*) \right) + iq \left( (D_\nu \Phi^*)(D^\nu \Phi) + \Phi^* (D_\nu D^\nu \Phi) \right)
\]

\[
= iq\Phi^* D^2 \Phi - iq\Phi D^2 \Phi^*
\]

\langle by equations of motion \rangle

\[
= iq\Phi^* (-m^2 \Phi) - iq\Phi (-m^2 \Phi^*)
\]

\[
= 0.
\] (S.9)

Q.E.D.

2
Problem 1(b):

According to the Noether theorem,

\[ T_{\text{Noether}}^{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu A_\lambda)} \partial^\nu A_\lambda + \frac{\partial L}{\partial (\partial_\mu \Phi)} \partial^\nu \Phi + \frac{\partial L}{\partial (\partial_\mu \Phi^*)} \partial^\nu \Phi^* - g^{\mu\nu} L \]  

(S.10)

\[ = T_{\text{Noether}}^{\mu\nu}(\text{EM}) + T_{\text{Noether}}^{\mu\nu}(\text{matter}) \]

where

\[ T_{\text{Noether}}^{\mu\nu}(\text{EM}) = -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \]  

(S.11)

similar to the free EM fields, and

\[ T_{\text{Noether}}^{\mu\nu}(\text{matter}) = D^\mu \Phi^* \partial^\nu \Phi + D^\mu \Phi \partial^\nu \Phi^* - g^{\mu\nu} (D^\lambda \Phi^* D_\lambda \Phi - m^2 \Phi^* \Phi). \]  

(S.12)

Both terms on the second line of eq. (S.10) lack \( \mu \leftrightarrow \nu \) symmetry and gauge invariance and thus need \( \partial_\lambda K^{\lambda\mu\nu} \) corrections for some \( K^{\lambda\mu\nu} = -K^{\mu\lambda\nu} \). We would like to show that the same \( K^{\lambda\mu\nu} = -F^{\mu\lambda} A^\nu \) we used to improve the free electromagnetic stress-energy tensor will now improve both the \( T_{\text{EM}}^{\mu\nu} \) and \( T_{\text{mat}}^{\mu\nu} \) at the same time!

Indeed, to improve the scalar fields' stress-energy tensor we need

\[ \Delta T_{\text{matter}}^{\mu\nu} = T_{\text{true}}^{\mu\nu}(\text{matter}) - T_{\text{Noether}}^{\mu\nu}(\text{matter}) \]

\[ = D^\mu \Phi^* (D^\nu \Phi - \partial^\nu \Phi) + D^\mu \Phi (D^\nu \Phi^* - \partial^\nu \Phi^*) \]

\[ = D^\mu \Phi^* (iq A^\nu \Phi) + D^\mu \Phi (-iq A^\nu \Phi^*) \]

\[ = -A^\nu (iq \Phi^* D^\mu \Phi - iq \Phi D^\mu \Phi^*) \]

\[ = -A^\nu J^\mu, \]  

(S.13)

while the improvement of the EM stress-energy requires (cf. previous homework 1.2)

\[ \Delta T_{\text{EM}}^{\mu\nu} = -F^{\mu\lambda} (F^\nu_\lambda - \partial^\nu A_\lambda) = +F^{\mu\lambda} \partial_\lambda A^\nu = \partial_\lambda (-F^{\lambda\mu} A^\nu) + A^\nu J^\mu. \]  

(S.14)

Altogether, to symmetrize the whole stress-energy tensor, we need

\[ \Delta T_{\text{tot}}^{\mu\nu} = T_{\text{true}}^{\mu\nu}(\text{total}) - T_{\text{Noether}}^{\mu\nu}(\text{total}) = \partial_\lambda \left( F^{\mu\lambda} A^\nu \equiv K^{\lambda\mu\nu} \right). \]  

Q.E.D.
Problem 1(c):

Because the fields $\Phi(x)$ and $\Phi^*(x)$ have opposite electric charges, their product is neutral and therefore $\partial_\mu(\Phi^*\Phi) = D_\mu(\Phi^*\Phi) = (D_\mu\Phi^*)\Phi + \Phi^*(D_\mu\Phi)$. Similarly,

$$
\partial_\mu((D^\mu\Phi^*) (D^{\nu}\Phi)) = (D_\mu D^\mu\Phi^*) (D^{\nu}\Phi) + (D^\mu\Phi^*) (D_\mu D^{\nu}\Phi) \\
= -m^2\Phi^* (D^{\nu}\Phi) + (D_\mu\Phi^*) (D^{\nu} D^\mu\Phi + iqF^{\mu\nu}\Phi)
$$

where we have applied the field equation $(D_\mu D^{\mu} + m^2)\Phi^*(x) = 0$ to the first term on the right hand side and used $[D_\mu, D^{\nu}]\Phi = iqF^{\mu\nu}\Phi$ to expand the second term. Likewise,

$$
\partial_\mu ((D^{\mu}\Phi) (D^{\nu}\Phi^*)) = (D_\mu D^{\mu}\Phi) (D^{\nu}\Phi^*) + (D^{\mu}\Phi) (D_\mu D^{\nu}\Phi^*) \\
= -m^2\Phi (D^{\nu}\Phi^*) + (D_\mu\Phi) (D^{\nu} D^\mu\Phi^* - iqF^{\mu\nu}\Phi^*)
$$

and

$$
\partial_\mu [-g^{\mu\nu} (D_\lambda\Phi^* D^{\lambda}\Phi - m^2\Phi^*\Phi)] = -\partial^{\nu} (D_\lambda\Phi^* D^{\lambda}\Phi) + m^2\partial^{\nu} (\Phi^*\Phi) \\
= - (D^{\nu} D^\mu\Phi^*) (D_\mu\Phi) - (D_\mu\Phi^*) (D^{\nu} D^\mu\Phi) \\
+ m^2\Phi (D^{\nu}\Phi^*) + m^2\Phi^* (D^{\nu}\Phi).
$$

Together, the left hand sides of eqs. (S.15), (S.16) and (S.17) comprise $\partial_\mu T_{\mu\nu}^{\text{mat}}$ — cf. eq. (7). On the other hand, combining the right hand sides of these three equations results in massive cancellation of all terms except those containing the gauge field strength tensor $F^{\mu\nu}$. Thus,

$$
\partial_\mu T_{\mu\nu}^{\text{mat}} = (D_\mu\Phi^*) (iqF^{\mu\nu}\Phi) + (D_\mu\Phi) (-iqF^{\mu\nu}\Phi^*) \\
= F^{\mu\nu} (iq\Phi D_\mu\Phi^* - iq\Phi^* D_\mu\Phi) \\
= F^{\mu\nu} J_\nu.
$$

And since in the previous homework we have shown $\partial_\mu T_{\mu\nu}^{\text{EM}} = -F^{\mu\nu} J_\nu$, it follows that the total stress tensor (4) is conserved, $\partial_\mu T^{\mu\nu} = 0$. \textit{Q.E.D.}
Problem 2(a):
Thanks to unitarity of the $U(x)$ matrix, the derivative of $U^\dagger(x) \equiv U^{-1}(x)$ is

$$\partial_\mu U^\dagger(x) = -U^\dagger(x) (\partial_\mu U(x)) U^\dagger(x). \quad (S.19)$$

Consequently, the ordinary derivative $\partial_\mu \Phi(x)$ of the adjoint field transforms under the local $U(x)$ symmetry into

$$\partial_\mu \Phi'(x) \equiv \partial_\mu \left(U(x)\Phi(x)U^\dagger(x)\right)$$

$$= (\partial_\mu U)\Phi U^\dagger + U(\partial_\mu \Phi)U^\dagger + U\Phi\left(\partial_\mu U^\dagger = -U^\dagger(\partial_\mu U)U^\dagger\right) \quad (S.20)$$

$$= U(\partial_\mu \Phi)U^\dagger + \left[(\partial_\mu U)U^\dagger, U\Phi U^\dagger\right].$$

At the same time, the vector field matrix $A_\mu(x)$ transforms as discussed in class,

$$A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + i(\partial_\mu U(x))U^\dagger(x). \quad (S.21)$$

Given these two formulae, the covariant derivative (12) transforms to

$$[D_\mu \Phi(x)]' = \partial_\mu \Phi'(x) + i[A'_\mu(x), \Phi'(x)]$$

$$= \partial_\mu(U\Phi U^\dagger) + i[U A_\mu U^\dagger, U\Phi U^\dagger] - \left[(\partial_\mu U)U^\dagger, U\Phi U^\dagger\right]$$

$$= U(\partial_\mu \Phi)U^\dagger + \left[(\partial_\mu U)U^\dagger, U\Phi U^\dagger\right] + iU[A_\mu, \Phi]U^\dagger - \left[(\partial_\mu U)U^\dagger, U\Phi U^\dagger\right]$$

$$= U(\partial_\mu \Phi + iA_\mu, \Phi)U^\dagger$$

$$\equiv U(x)(D_\mu \Phi(x))U^\dagger(x). \quad (S.22)$$

In other words, the covariant derivative defined by eq. (12) is indeed covariant. \textit{Q.E.D.}

Problem 2(b):

$$D_\mu D_\nu \Phi = D_\mu (\partial_\nu \Phi + i[A_\nu, \Phi]) = \partial_\mu (\partial_\nu \Phi + i[A_\nu, \Phi]) + i[A_\mu, (\partial_\nu \Phi + i[A_\nu, \Phi])]$$

$$= \partial_\mu \partial_\nu \Phi + i[(\partial_\mu A_\nu), \Phi] + i[A_\nu, \partial_\mu \Phi] + i[A_\mu, \partial_\nu \Phi] - [A_\mu, [A_\nu, \Phi]]. \quad (S.23)$$

Similarly,

$$D_\nu D_\mu \Phi = \partial_\nu \partial_\mu \Phi + i[(\partial_\nu A_\mu), \Phi] + i[A_\mu, \partial_\nu \Phi] + i[A_\nu, \partial_\mu \Phi] - [A_\nu, [A_\mu, \Phi]]. \quad (S.24)$$

Three out of five terms on the right hand sides of these formulæ are identical and hence cancel
out of the difference \( D_\mu D_\nu \Phi - D_\nu D_\mu \Phi \). The remaining terms comprise the commutator

\[
[D_\mu, D_\nu] \Phi = i[(\partial_\mu A_\nu), \Phi] - i[(\partial_\nu A_\mu), \Phi] - [A_\mu, [A_\nu, \Phi]] + [A_\nu, [A_\mu, \Phi]] \\
= i[(\partial_\mu A_\nu - \partial_\nu A_\mu), \Phi] - [[A_\mu, A_\nu], \Phi] \\
\equiv i[F_{\mu\nu}, \Phi].
\]

(S.25)

**Problem 2(c):**

First, let us evaluate

\[
D_\lambda F_{\mu\nu} = \partial_\lambda (\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]) + i[A_\lambda, \partial_\lambda (\partial_\mu A_\nu - \partial_\nu A_\mu)] \\
= \left( \partial_\lambda \partial_\mu A_\nu - \partial_\lambda \partial_\nu A_\mu \right) + i \left( [A_\mu, \partial_\lambda A_\nu] - [A_\lambda, \partial_\mu A_\nu] \right) \\
+ i \left( [A_\lambda, \partial_\mu A_\nu] - [A_\mu, \partial_\nu A_\mu] \right) - [A_\lambda, [A_\mu, A_\nu]].
\]

(S.26)

For each group of terms here, summing over cyclic permutation of the Lorentz indices \( \lambda \rightarrow \mu \rightarrow \nu \rightarrow \lambda \) produces a zero:

\[
(\partial_\lambda \partial_\mu A_\nu - \partial_\lambda \partial_\nu A_\mu) + (\partial_\nu \partial_\lambda A_\mu - \partial_\mu \partial_\lambda A_\nu) + (\partial_\mu \partial_\lambda A_\nu - \partial_\gamma \partial_\lambda A_\mu) = 0,
\]

\[
([A_\mu, \partial_\lambda A_\nu] - [A_\nu, \partial_\lambda A_\mu]) + ([A_\nu, \partial_\mu A_\lambda] - [A_\lambda, \partial_\mu A_\nu]) + ([A_\lambda, \partial_\nu A_\mu] - [A_\mu, \partial_\nu A_\lambda]) = 0,
\]

\[
([A_\lambda, \partial_\mu A_\nu] - [A_\mu, \partial_\nu A_\lambda]) + ([A_\mu, \partial_\nu A_\lambda] - [A_\lambda, \partial_\mu A_\nu]) + ([A_\nu, \partial_\lambda A_\mu] - [A_\mu, \partial_\lambda A_\nu]) = 0,
\]

and consequently

\[
D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0.
\]

(S.27)

**Alternative Solution:**

Suppose \( \phi(x) \) is a fundamental multiplet of fields, *i.e.* a column vector of \( N \) complex fields transforming according to \( \psi'(x) = U(x)\psi(x) \). Then \( [D_\mu, D_\nu] \phi(x) = i F_{\mu\nu}(x) \times \phi(x) \) and hence

\[
[D_\lambda, [D_\mu, D_\nu]] \phi(x) = i[D_\lambda, F_{\mu\nu}(x) \times \phi(x)] = i(D_\lambda F_{\mu\nu}(x)) \times \phi(x),
\]

(S.28)

6
where the second equality follow from the Leibniz rule

\[ D_\lambda(F_{\mu\nu} \times \phi) = D_\lambda F_{\mu\nu} \times \phi + F_{\mu\nu} \times D_\lambda \phi. \]  

(S.29)

Consequently, for any \( \phi(x) \),

\[ i \left( D_\lambda F_{\mu\nu}(x) + D_\mu F_{\nu\lambda}(x) + D_\nu F_{\lambda\mu}(x) \right) \times \phi(x) = \]

\[ = [D_\lambda[D_\mu, D_\nu]]\phi(x) + [D_\mu, [D_\nu, D_\lambda]]\phi(x) + [D_\nu, [D_\lambda, D_\mu]]\phi(x) = 0 \]

because of Jacobi identity for the commutators: for any linear operators \( A, B, \) and \( C, [A, [B, C]] + [B, [A, C]] + [C, [A, B]] = 0 \) and in particular

\[ [D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] = 0. \]  

(S.31)

And since eq. (S.30) holds for any \( \phi(x) \), it implies

\[ D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0. \]  

(9)

Problem 2(d):

\[ \delta (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]) = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu + i[\delta A_\mu, A_\nu] + i[A_\mu, \delta A_\nu] \]

\[ = D_\mu \delta A_\nu - D_\nu \delta A_\mu \]

(S.32)

Problem 2(e): Working out the Euler–Lagrange equations for the vector fields \( A_\mu(x) = \sum_a A^a_\mu(x) \frac{\lambda}{2} \) is rather painful and it’s easier to directly evaluate the action variation \( \delta S \).

In light of the previous question (d), we have

\[ \delta \text{tr} \left( F^{\mu\nu} F_{\mu\nu} \right) = 2 \text{tr} \left( F^{\mu\nu} \delta F_{\mu\nu} \right) = 2 \text{tr} \left( F^{\mu\nu} (D_\mu \delta A_\nu - D_\nu \delta A_\mu) \right) \]

\[ = 4 \text{tr} \left( F^{\mu\nu} D_\mu \delta A_\nu \right) = 4 \partial_\mu \text{tr} \left( F^{\mu\nu} \delta A_\nu \right) - 4 \text{tr} \left( \delta A_\nu \times D_\mu F^{\mu\nu} \right) \]

\[ = \text{total derivative} - 2 \sum_a \delta A^a_\nu \times (D_\mu F^{\mu\nu})^a \]

(S.33)
where the last equality follows from

$$\delta A_\nu = \sum_a \delta A_{\mu}^a \frac{\lambda^a}{2}, \quad D_\mu F^{\mu\nu} = \sum_b (D_\mu F^{\mu\nu})^b \frac{\lambda^b}{2}, \quad \text{and} \quad \text{tr} \left( \frac{\lambda^a}{2} \frac{\lambda^b}{2} \right) = \frac{\delta^{ab}}{2}. \quad (S.34)$$

Consequently, the Yang–Mills action

$$S = -\frac{1}{2g^2} \int d^4x \text{tr} \left( F^{\mu\nu} F_{\mu\nu} \right) \quad (S.35)$$

varies by

$$\delta S = \frac{1}{g^2} \int d^4x \sum_a \delta A^a_\nu (x) \times (D_\mu F^{\mu\nu})^a (x), \quad (S.36)$$

and demanding that $\delta S = 0$ for any $\delta A^a_\mu (x)$ gives us the Yang–Mills field equations

$$D_\mu F^{\mu\nu} (x) = 0. \quad (S.37)$$