
Problem 1(a):
For \( \mathbf{p} = 0 \), \( p^0 = +m \) and \( p^0 - m = m(\gamma^0 - 1) \). Hence, \( u(\mathbf{p} = 0, s) \) satisfy \((\gamma^0 - 1)u = 0\), or in the Weyl basis

\[
\begin{pmatrix}
-1_{2 \times 2} & 1_{2 \times 2} \\
1_{2 \times 2} & -1_{2 \times 2}
\end{pmatrix}
\begin{pmatrix}
u
\end{pmatrix}
= 0 \quad \Rightarrow \quad u = \begin{pmatrix}
\zeta \\
\zeta
\end{pmatrix}
\]

(S.1)

where \( \zeta \) is an arbitrary two-component spinor. It’s normalization follows from \( u^\dagger u = 2\zeta^\dagger\zeta \), so we want \( \zeta^\dagger\zeta = E_p = m \). In other words, \( \zeta = \sqrt{m} \times \xi \) — and hence \( u \) is as in eq. (2) — where \( \xi^\dagger\xi = 1 \).

There are two independent choices of \( \xi \), normalized to \( \xi_s^\dagger\xi_{s'} = \delta_{s,s'} \) so that

\[
u^\dagger(0, s)u(0, s') = 2m\delta_{s,s'} = 2E_p\delta_{s,s'}.
\]

They correspond to two spin states of a \( \mathbf{p} = 0 \) electron. Indeed, in the Weyl basis where the spin matrices are

\[
S = \frac{1}{2}
\begin{pmatrix}
\sigma & 0 \\
0 & \sigma
\end{pmatrix}
\]

(cf. problem 8.4(a) of the previous homework set), we have

\[
u^\dagger(0, s) S u(0, s') = 2m \times \xi_{s'}^\dagger \frac{\sigma}{2} \xi_{s'}. \quad (S.2)
\]

Problem 1(b):
Dirac equation is Lorentz covariant, so we may obtain solutions for all \( p^\mu = (+E_p, \mathbf{p}) \) by simply Lorentz-boosting the solutions (2) for \( p^\mu = (+m, \mathbf{0}) \),

\[
u(p, s) = M \nu(0, s) = \begin{pmatrix}
M_L & 0 \\
0 & M_R
\end{pmatrix}
\begin{pmatrix}
\sqrt{m} \xi_s \\
\sqrt{m} \xi_s
\end{pmatrix}
= \begin{pmatrix}
\sqrt{m} M_L \xi_s \\
\sqrt{m} M_R \xi_s
\end{pmatrix}
\]

(S.3)

where \( M, M_L, \) and \( M_R \) are respectively Dirac-spinor, LH–Weyl-spinor, and RH–Weyl-spinor representations or the Lorentz boost \((m, \mathbf{0}) \mapsto (E, \mathbf{p}) \). In problem 8.4.(c) of the previous homework
we saw that
\[ M_L = \sqrt{\gamma} \times \sqrt{1 - \beta n \cdot \sigma}, \quad M_R = \sqrt{\gamma} \times \sqrt{1 + \beta n \cdot \sigma}, \tag{S.4} \]
where \( \beta n \) is the velocity vector of the boost (in units of \( c \)) and \( \gamma = 1/\sqrt{1 - \beta^2} \). For the boost we are interested in,
\[ \gamma = \frac{E}{m}, \quad \gamma \beta n = \frac{p}{m}, \tag{S.5} \]
so
\[ m \times \gamma \times (1 \mp \beta n \cdot \sigma) = E \mp p \cdot \sigma \tag{S.6} \]
and hence
\[ \sqrt{m} M_L = \sqrt{E - p \cdot \sigma}, \quad \sqrt{m} M_R = \sqrt{E + p \cdot \sigma}. \tag{S.7} \]
Plugging these matrices into eq. (S.3) gives us
\[ u(p, s) = \begin{pmatrix} \sqrt{E - p \cdot \sigma \xi_s} \\ \sqrt{E + p \cdot \sigma \xi_s} \end{pmatrix}. \tag{3} \]
\[ Q.E.D. \]

Problem 1(c):
\[ v(p, s) = \gamma^2 u^*(p, s) = \begin{pmatrix} 0 & +\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \times \begin{pmatrix} \sqrt{E - p \cdot \sigma \xi_s}^* \\ \sqrt{E + p \cdot \sigma \xi_s}^* \end{pmatrix} \]
\[ = \begin{pmatrix} +\sigma_2 & (\sqrt{E + p \cdot \sigma})^* \xi_s^* \\ -\sigma_2 & (\sqrt{E - p \cdot \sigma})^* \xi_s^* \end{pmatrix} \]
\[ = \begin{pmatrix} +\sigma_2 & \sigma_2 \times \eta_s \\ -\sigma_2 & \sigma_2 \times \eta_s \end{pmatrix} \quad \text{where} \quad \eta_s \overset{\text{def}}{=} \sigma_2 \times \xi_s^* \]
\[ = \begin{pmatrix} +\sqrt{E - p \cdot \sigma} \times \eta_s \\ -\sqrt{E + p \cdot \sigma} \times \eta_s \end{pmatrix} \]
because \( \sigma_2 \sigma^* \sigma_2 = -\sigma \quad \implies \quad \sigma^2 \left( \sqrt{E \pm p \cdot \sigma} \right)^* \sigma_2 = \sqrt{E \mp p \cdot \sigma}. \tag{S.8} \]
This verifies eq. (4) for the negative-energy plane waves.
To find the 3D spin of the $\eta$, 3D spinors, we calculate
\[ \eta^\dagger \sigma \eta = \xi^\dagger \sigma_2 \sigma_2 \xi^* = \xi^\dagger (-\sigma^*) \xi^* = -\left( \xi^\dagger \sigma \xi \right)^* = -\xi^\dagger \sigma \xi, \]  
(S.9)
where the first equality follows from the definition $\eta \overset{\text{def}}{=} \sigma_2 \times \xi^*$, and the last equality follows from hermiticity of the Pauli matrices (which makes the $\xi^\dagger \sigma \xi$ vector real). In 3D, $S = \frac{1}{2} \sigma$, so we have just proved
\[ \eta^\dagger S \eta = -\xi^\dagger S \xi, \]  
(S.10)
which means that $\eta$ has precisely opposite spin state from $\xi$.

Problem 1(d):
3D spinors $\xi_\lambda$ of definite helicity $\lambda = \mp \frac{1}{2}$ satisfy
\[ (p \cdot \sigma) \xi_\mp = \mp |p| \times \xi_\mp. \]  
(S.11)
Therefore, positive-energy Dirac spinors (3) of definite helicity may be written as
\[ u(p, \lambda = \mp \frac{1}{2}) = \left( \sqrt{E \pm |p|} \times \xi_\mp \right), \]  
(S.12)
In the ultra-relativistic limit $|p| \gg m$, we have $E \approx |p|$ and hence $\sqrt{E + |p|} \approx \sqrt{2E}$; by comparison, $\sqrt{E - |p|} \approx 0$. Consequently, eq. (S.12) simplifies to
\[ u(p, L) \approx \sqrt{2E} \left( \begin{array}{c} \xi_L \\ 0 \end{array} \right), \quad u(p, R) \approx \sqrt{2E} \left( \begin{array}{c} 0 \\ \xi_R \end{array} \right). \]  
(S.13)
In other words, positive-energy ultra-relativistic Dirac spinors of definite helicity are chiral — LH Weyl components only for left helicity and or RH Weyl components only for right helicity.
Now consider negative-energy Dirac spinors (4). Because the \( \eta_s \) spinors have exactly opposite spin from the \( \xi_s \), their helicities are also opposite and hence

\[
(p \cdot \sigma) \eta_\mp = \pm |p| \times \eta_\mp \tag{S.14}
\]

— note opposite sign from eq. (S.11). Therefore, negative-energy Dirac spinors \( v \) of definite helicity are

\[
v(p, \lambda = \mp \frac{1}{2}) = \left( \frac{+\sqrt{E \mp |p| \times \eta}}{\mp E \pm |p| \times \eta} \right),
\]

and in the ultra-relativistic limit they become

\[
v(p, L) \approx -\sqrt{2E} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, \quad v(p, R) \approx +\sqrt{2E} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}.
\]

\[
\tag{S.16}
\]

Again, the ultra-relativistic spinors are chiral, but this time chirality is opposite from helicity — LH Weyl components only for the right helicity and RH Weyl components only for the left helicity.

**Problem 1(e):**

Given \( u(p, s) \) spinors as in eq. (3), we have

\[
u^\dagger(p, s)u(p, s') = \xi_s^\dagger \left( (\sqrt{E - p \cdot \sigma})^2 + (\sqrt{E + p \cdot \sigma})^2 \right) \xi_{s'} = \xi_s^\dagger (2E) \xi_{s'} = 2E \delta_{s,s'}.
\]

\[
\tag{S.17}
\]

Likewise, for the \( v(p, s) \) spinors as in eq. (4), we have

\[
u^\dagger(p, s)v(p, s') = \eta_s^\dagger \left( (+\sqrt{E - p \cdot \sigma})^2 + (-\sqrt{E + p \cdot \sigma})^2 \right) \eta_{s'} = \eta_s^\dagger (+2E) \eta_{s'} = 2E \delta_{s,s'}
\]

\[
\tag{S.18}
\]

because \( \eta_s^\dagger \eta^s_{s'} = \xi_s^\dagger \sigma_2 \sigma_2 \xi_{s'}^* = (\xi_s^\dagger \xi_{s'})^* = \delta_{s,s'} \).
Now consider the Lorentz invariant products $\bar{u}u$ and $\bar{v}v$. The $\bar{u}$ and $\bar{v}$ are given by

$$\bar{u}(p, s) = u^\dagger(p, s)\gamma^0 = \left(\begin{array}{c} \sqrt{E - p \cdot \sigma} \\
\sqrt{E + p \cdot \sigma} \end{array}\right) \left(\begin{array}{cc} 0 & 1_{2 \times 2} \\
1_{2 \times 2} & 0 \end{array}\right)$$

$$= (\xi_s \times \sqrt{E + p \cdot \sigma}, \xi_s \times \sqrt{E - p \cdot \sigma}),$$

(S.19)

$$\bar{v}(p, s) = v^\dagger(p, s)\gamma^0 = \left(\begin{array}{c} +\sqrt{E - p \cdot \sigma} \eta_s \\
-\sqrt{E + p \cdot \sigma} \eta_s \end{array}\right) \left(\begin{array}{cc} 0 & 1_{2 \times 2} \\
1_{2 \times 2} & 0 \end{array}\right)$$

$$= (-\eta_s \times \sqrt{E + p \cdot \sigma}, +\eta_s \times \sqrt{E - p \cdot \sigma}).$$

Consequently,

$$\bar{u}(p, s) u(p, s') = \xi_s \times \sqrt{E + p \cdot \sigma} \times \sqrt{E - p \cdot \sigma} \times \xi_{s'}$$

$$+ \xi_s \times \sqrt{E - p \cdot \sigma} \times \sqrt{E + p \cdot \sigma} \times \xi_{s'}$$

(S.20)

$$= 2m \times \xi_s^\dagger \xi_{s'} = 2m\delta_{s,s'}$$

because

$$\sqrt{E - p \cdot \sigma} \times \sqrt{E - p \cdot \sigma} = \sqrt{E - p \cdot \sigma} \times \sqrt{E + p \cdot \sigma} = \sqrt{E^2 - (p \cdot \sigma)^2} = \sqrt{E^2 - p^2} = m.$$  

(S.21)

Likewise,

$$\bar{v}(p, s) v(p, s') = -\eta_s \times \sqrt{E + p \cdot \sigma} \times \sqrt{E - p \cdot \sigma} \times \eta_{s'}$$

$$- \eta_s \times \sqrt{E - p \cdot \sigma} \times \sqrt{E + p \cdot \sigma} \times \eta_{s'}$$

(S.22)

$$= -2m \times \eta_s^\dagger \eta_{s'} = -2m\delta_{s,s'}.$$

Problem 1(f): In matrix notations (column $\times$ row = matrix), we have

$$u(p, s) \times \bar{u}(p, s) = \left(\begin{array}{c} \sqrt{E - p \sigma} \xi_s \\
\sqrt{E + p \sigma} \xi_s \end{array}\right) \times \left(\begin{array}{c} \xi_s^\dagger \sqrt{E + p \sigma}, \xi_s^\dagger \sqrt{E - p \sigma} \\
\xi_s^\dagger \sqrt{E + p \sigma}, \xi_s^\dagger \sqrt{E - p \sigma} \end{array}\right)$$

(S.23)

$$= \left(\begin{array}{cc} \sqrt{E - p \sigma} \xi_s \times \xi_s^\dagger \sqrt{E + p \sigma} & \sqrt{E - p \sigma} \xi_s \times \xi_s^\dagger \sqrt{E - p \sigma} \\
\sqrt{E + p \sigma} \xi_s \times \xi_s^\dagger \sqrt{E + p \sigma} & \sqrt{E + p \sigma} \xi_s \times \xi_s^\dagger \sqrt{E - p \sigma} \end{array}\right).$$
Summing over two spin polarizations replaces \( \xi_s \times \xi_s^\dagger \) with \( \sum_s \xi_s \times \xi_s^\dagger = 1_{2\times 2} \). Consequently,

\[
\sum_s u(p, s) \times \bar{u}(p, s) = \\
= \left( \sqrt{E - p\sigma} \left[ \sum_s \xi_s \times \xi_s^\dagger \right] \sqrt{E + p\sigma} \right) \left( \sqrt{E - p\sigma} \left[ \sum_s \xi_s \times \xi_s^\dagger \right] \sqrt{E + p\sigma} \right) \\
= \left( \sqrt{E - p\sigma} \times \sqrt{E + p\sigma} \right) \left( \sqrt{E - p\sigma} \times \sqrt{E + p\sigma} \right) \\
= \left( \begin{array}{c}
m \ E - p\sigma \\
E + p\sigma & m \end{array} \right) = m \times 1_{4\times 4} + E \times \gamma^0 - p \cdot \vec{\gamma} \\
= \not{p} + m.
\]

Similarly

\[
v(p, s) \times \bar{v}(p, s) = \left( \begin{array}{c}
+ \sqrt{E - p\sigma} \eta_s \\
- \sqrt{E + p\sigma} \eta_s \end{array} \right) \times \left( \begin{array}{c}
- \eta_s^\dagger \sqrt{E + p\sigma}, + \eta_s^\dagger \sqrt{E - p\sigma} \\
- \eta_s^\dagger \sqrt{E - p\sigma}, + \eta_s^\dagger \sqrt{E + p\sigma} \end{array} \right) \\
= \left( \begin{array}{c}
- \sqrt{E - p\sigma} \left( \eta_s \times \eta_s^\dagger \right) \sqrt{E + p\sigma} + \sqrt{E - p\sigma} \left( \eta_s \times \eta_s^\dagger \right) \sqrt{E - p\sigma} \\
+ \sqrt{E + p\sigma} \left( \eta_s \times \eta_s^\dagger \right) \sqrt{E - p\sigma} - \sqrt{E + p\sigma} \left( \eta_s \times \eta_s^\dagger \right) \sqrt{E + p\sigma} \\
\end{array} \right),
\]

where \( \eta_s \times \eta_s^\dagger = \sigma_2 \left( \xi_s^\dagger \times \xi_s \right)^* \sigma_2 \) and hence

\[
\sum_s \eta_s \times \eta_s^\dagger = \sigma_2 \left( \sum_s \xi_s^\dagger \times \xi_s \right)^* \sigma_2 = \sigma_2 1_{2\times 2}^* \sigma_2 = 1_{2\times 2}.
\]

Therefore,

\[
\sum_s v(p, s) \times \bar{v}(p, s) = \left( \begin{array}{c}
- \sqrt{E - p\sigma} \times 1_{2\times 2} \times \sqrt{E + p\sigma} + \sqrt{E - p\sigma} \times 1_{2\times 2} \times \sqrt{E - p\sigma} \\
+ \sqrt{E + p\sigma} \times 1_{2\times 2} \times \sqrt{E + p\sigma} - \sqrt{E + p\sigma} \times 1_{2\times 2} \times \sqrt{E - p\sigma} \\
\end{array} \right) \\
= \left( \begin{array}{c}
- \not{m} \ E - p\sigma \\
E + p\sigma & - \not{m} \end{array} \right) = - \not{m} \times 1_{4\times 4} + E \times \gamma^0 - p \cdot \vec{\gamma} \\
= \not{p} - m.
\]

\textit{Q.E.D.}
Problem 1(g):
The constant spinors $u \equiv u(p, s)$ and $\bar{u}' \equiv \bar{u}(p', s')$ satisfy Dirac equations $\not{p}u = mu$ and $\bar{u}' \not{p}' = m\bar{u}'$. Applying both equations to the Dirac “sandwich” $\bar{u}' \gamma^\mu u$, we have

$$\bar{u}' \gamma^\mu u = \frac{1}{m} \bar{u}' \not{p}' \times \gamma^\mu u = \frac{1}{m} \bar{u}' \gamma^\mu \times \not{p} u = \frac{1}{2m} \bar{u}' (\not{p}' \gamma^\mu + \gamma^\mu \not{p}) u.$$ (S.28)

On the right hand side, we evaluate

$$\not{p}' \gamma^\mu + \gamma^\mu \not{p} \equiv \not{p}' \gamma^\nu \gamma^\mu + p_\nu \gamma^\mu \gamma^\nu = \not{p}_\nu (g^{\mu\nu} + 2iS^{\mu\nu}) + p_\nu (g^{\mu\nu} - 2iS^{\mu\nu})$$

$$= (p' + p)_\nu \times g^{\mu\nu} + 2i(p' - p)_\nu \times S^{\mu\nu},$$ (S.29)

which gives us the Gordon identity

$$\bar{u}' \gamma^\mu u = \frac{(p' + p)^\mu}{2m} \bar{u}' u + \frac{i(p' - p)_\nu}{m} \bar{u}' S^{\mu\nu} u.$$ (3)

Q.E.D.

Problem 1(h):
The negative-frequency spinors $v \equiv v(p, s)$ and $\bar{v}' \equiv \bar{v}(p', s')$ satisfy Dirac equations $\not{p}v = -mv$ and $\bar{v}' \not{p}' = -m\bar{v}'$. Consequently, proceeding exactly as above modulo signs, we have

$$\bar{v}' \gamma^\mu v = \frac{(p' - p)^\mu}{2m} \bar{v}' v + \frac{i(p' + p)_\nu}{m} \bar{v}' S^{\mu\nu} v,$$

$$\bar{v}' \gamma^\mu u = \frac{(-p' + p)^\mu}{2m} \bar{v}' u + \frac{i(-p' - p)_\nu}{m} \bar{v}' S^{\mu\nu} u,$$ (S.30)

$$\bar{v}' \gamma^\mu v = \frac{(-p' - p)^\mu}{2m} \bar{v}' v + \frac{i(-p' + p)_\nu}{m} \bar{v}' S^{\mu\nu} v.$$
Problem 2(a):
Despite anticommutativity of the fermionic fields, the Hermitian conjugation of an operator product reverses the order of operators without any extra sign factors, thus \((\Psi_\alpha \psi_\beta)^\dagger = +\psi_\beta \Psi_\alpha\).
Consequently, for any \(4 \times 4\) matrix \(\Gamma\), \((\Psi^\dagger \Gamma \Psi)^\dagger = +\Psi^\dagger \Gamma^\dagger \Psi\), and hence \((\overline{\psi} \Gamma \psi)^\dagger = \overline{\psi} \Gamma^\dagger \psi\) where \(\Gamma = \gamma^0 \Gamma^\dagger \gamma^0\) is the Dirac conjugate of \(\Gamma\).

Now consider the 16 matrices which appear in the bilinears (1). Obviously \(\overline{1} = +1\) and this gives us \(S^\dagger = +S\). We saw in class that \(\gamma^\mu = +\gamma^\mu\), and this gives us \((V^\mu)^\dagger = +V^\mu\). We also saw that \(i\gamma^{[\mu} \gamma^{\nu]} = -i\gamma^{[\nu} \gamma^{\mu]} = +i\gamma^{[\mu} \gamma^{\nu]}\), and this gives us \((T^{\mu\nu})^\dagger = +T^{\mu\nu}\). As to the \(\gamma^5\) matrix, we saw in homework 8.4(b) that it’s Hermitian and anticommutes with all \(\gamma^0\). Hence \(\gamma^5 = \gamma^5 \gamma^0 = +\gamma^5 \gamma^0 = -\gamma^5\) which gives us \(P^\dagger = +P\). Finally, \(\gamma^5 \gamma^\mu = \gamma^5 \gamma^\mu = -\gamma^\mu \gamma^5 = +\gamma^5 \gamma^\mu\), which gives us \((A^\mu)^\dagger = +A^\mu\). Thus, by inspection, all the bilinears (10) are Hermitian. \(\text{Q.E.D.}\)

Problem 2(b):
Under a continuous Lorentz symmetry \(x \mapsto x' = Lx\), the Dirac spinor field and its conjugate transform according to
\[
\begin{align*}
\Psi'(x') &= M(L) \Psi(x = L^{-1} x'), \\
\overline{\Psi}'(x') &= \overline{\Psi}(x = L^{-1} x') M^{-1}(L),
\end{align*}
\]  
(S.31)
hence any bilinear \(\overline{\psi} \Gamma \psi\) transforms according to
\[
\overline{\Psi}'(x') \Gamma \Psi(x') = \overline{\Psi}(x) \Gamma' \Psi(x)
\]  
(S.32)
where
\[
\Gamma' = M^{-1}(L) \Gamma M(L).
\]  
(S.33)
Obviously for \(\Gamma = 1\), \(\Gamma' = M^{-1} M = 1\), which makes \(S\) a Lorentz scalar. According to to homework problem 8.2(b), for \(\Gamma = \gamma^\mu\), \(\Gamma' = M^{-1} \gamma^\mu M = L^\nu_\mu \gamma^\nu\); this makes \(V^\mu\) a Lorentz vector.

For \(\Gamma = \gamma^\mu \gamma^\nu\), \(M^{-1} \gamma^\mu \gamma^\nu M = (M^{-1} \gamma^\mu M)(M^{-1} \gamma^\nu M) = L^\mu_\kappa \gamma^\kappa \times L^\nu_\lambda \gamma^\lambda\). Consequently for \(\Gamma = \gamma^{[\mu} \gamma^{\nu]}\), \(\Gamma' = L^\mu_\kappa L^\nu_\lambda \gamma^{[\kappa} \gamma^{\lambda]}\), which makes \(T^{\mu\nu}\) a Lorentz tensor (with two antisymmetric indices).
Finally, the $\gamma^5$ commutes with even products of the $\gamma^\mu$ matrices and hence with $M(L) = \exp\left(\frac{1}{4} \Theta_{\mu\nu} \gamma^\mu \gamma^\nu\right)$. Consequently, $M^{-1} \gamma^5 M = \gamma^5$, which makes $P$ a Lorentz scalar. Likewise, $M^{-1}(\gamma^\mu \gamma^5) M = (M^{-1} \gamma^\mu M) \gamma^5 = L^\mu \gamma^\nu \gamma^5$, which makes $A^\mu$ a Lorentz vector. \textit{Q.E.D.}

**Problem 2(c):**
In class I showed that under the parity symmetry $P : (x, t) = (-x, +t)$, the Dirac fields transform as

$$\Psi'(x') = \pm \gamma^0 \Psi(x), \quad \overline{\Psi}'(x') = \pm \overline{\Psi}(x) \gamma^0. \quad (S.34)$$

Consequently, the Dirac bilinears transform as

$$\mathcal{P} : \overline{\Psi'} \Gamma \Psi' \bigg|_x \mapsto \overline{\Psi'} \Gamma \Psi' \bigg|_{x'} = \overline{\Psi'} \gamma^0 \Gamma \gamma^0 \Psi \bigg|_x. \quad (S.35)$$

By inspection, out of 16 possible $\Gamma$ matrices, $1$, $\gamma^0$, $\gamma^i \gamma^j$, and $\gamma^5 \gamma^i$ commute with the $\gamma^0$, while $\gamma^i$, $\gamma^0 \gamma^i$, $\gamma^5 \gamma^0$, and $\gamma^5$ anticommute with the $\gamma^0$. Therefore,

- the $S$, $V^0$, $T^{ij}$, and $A^i$ remain invariant under parity, while
- the $V^i$, $T^{0i}$, $A^0$, and $P$ change their signs.

From the 3D point of view, this means that $S$ and $V^0$ are true scalars, $P$ and $A^0$ are pseudoscalars, $V$ is a true or polar vector, $A$ is a pseudovector or axial vector, and the tensor $T$ contains one true vector $T^{0i}$ and one axial vector $\frac{1}{2} \epsilon^{ijk} T^{jk}$. In space-time terms, we call $S$ a true (Lorentz) scalar, $P$ a (Lorentz) pseudoscalar, $V^\mu$ a true (Lorentz) vector, and $A^\mu$ an axial (Lorentz) vector. Pedantically speaking, $T^{\mu\nu}$ is a true Lorentz tensor while $\tilde{T}^{\kappa\lambda} \equiv \frac{1}{2} \epsilon^{\kappa\lambda\mu\nu} T_{\mu\nu}$ is a Lorentz pseudotensor, but few people are that pedantic.

**Problem 2(d):**
In the Weyl convention for the Dirac matrices, the charge conjugation symmetry $\mathcal{C}$ acts on the Dirac field according to $\Psi'(x) = \pm \gamma^2 \Psi^*(x)$ and hence

$$\Psi'^\dagger(x) = \mp \Psi^\dagger(x) \gamma^2 \quad \Rightarrow \quad \overline{\Psi'}(x) = \Psi'^\dagger(x) \gamma^0 = \mp \Psi^\dagger(x) \gamma^2 \gamma^0. \quad (S.36)$$

Consequently, for any Dirac bilinear $\overline{\Psi} \Gamma \Psi$,

$$\overline{\Psi}' \Gamma \Psi' = -\overline{\Psi}^\dagger \gamma^2 \gamma^0 \Gamma \gamma^2 \Psi^* = + \Psi^\dagger (\gamma^2 \gamma^0 \Gamma \gamma^2)^\dagger \Psi = + \overline{\Psi} \gamma^0 \gamma^2 \Gamma^\dagger \gamma^0 \gamma^2 \Psi \equiv \overline{\Psi} \Gamma^c \Psi. \quad (S.37)$$
The second equality here follows by transposition of the Dirac “sandwich” $\Psi^\top \cdots \Psi^*$, which carries an extra minus sign because the fermionic fields $\Psi$ and $\Psi^*$ anticommute with each other. The third equality follows from $(\gamma^0)^\top = +\gamma^0$, $(\gamma^2)^\top = +\gamma^2$, and $\Psi^\dagger = \bar{\Psi}\gamma^0$.

Problem 2(e):

By inspection, $1^c \equiv \gamma^0\gamma^2\gamma^0\gamma^2 = +1$. The $\gamma_5$ matrix is symmetric and commutes with the $\gamma^0\gamma^2$, hence $\gamma_5^c = +\gamma_5$. Among the four $\gamma_\mu$ matrices, the $\gamma_1$ and $\gamma_3$ are anti-symmetric and commute with the $\gamma^0\gamma^2$ while the $\gamma_0$ and $\gamma_2$ are symmetric but anti-commute with the $\gamma^0\gamma^2$; hence, for all four $\gamma_\mu$, $\gamma_\mu^c = -\gamma_\mu$. Finally, because of the transposition involved, $(\gamma_\mu\gamma_\nu)^c = \gamma_\nu^c\gamma_\mu^c = +\gamma_\nu\gamma_\mu$, hence $(\gamma^{[\mu\nu]}_\mu)^c = +\gamma^{[\nu\mu]}_\mu = -\gamma^{[\mu\nu]}$. Likewise, $(\gamma^5\gamma_\mu)^c = (\gamma_\mu)^c(\gamma^5)^c = -\gamma^\mu\gamma_5 = +\gamma^5\gamma_\mu$.

Therefore, according to eq. (S.37), the scalar $S$, the pseudoscalar $P$, and the axial vector $A_\mu$ are $C$–even, while the vector $V_\mu$ and the tensor $T_{\mu\nu}$ are $C$–odd.

Problem 2(f):

As discussed in class, in the Weyl fermion language, the Lagrangian for $\chi$, $\bar{\chi}$ and their hermitian conjugates is obviously invariant under the charge conjugation $C : \chi \leftrightarrow \bar{\chi}$. In the Dirac-spinor language, this is less obvious, hence this exercise.

In the previous question we saw that $\bar{\Psi}\Psi$ is $C$–even, which means that the mass term $-m\bar{\Psi}\Psi$ is $C$–invariant. For the rest of the Lagrangian, we have

$$C : \bar{\Psi}\partial\Psi \mapsto -\Psi^\top\gamma^2\gamma^0\gamma^\mu\partial_\mu\gamma^2\Psi^*$$

$$= + (\partial_\mu\Psi^\dagger) (\gamma^2\gamma^0\gamma^\mu\gamma^2)^\top \Psi$$

$$= + (\partial_\mu\bar{\Psi}) \gamma^0\gamma^2 (\gamma^\mu)^\top \gamma^0\gamma^2\Psi$$

$$= - (\partial_\mu\bar{\Psi}) \gamma^\mu\Psi$$

$$= +\bar{\Psi}\partial\Psi - \partial_\mu (\bar{\Psi}\gamma^\mu\Psi).$$

(S.38)

The second line here follows by transposition of the Dirac ‘sandwich’ $\Psi^\top \cdots \partial_\mu\Psi^*$, and the sign changes because of Fermi statistics ($\Psi_\alpha$ and $\Psi^*_\beta$ anticommute). The fourth line follows from

$$\gamma^0\gamma^2 (\gamma^\mu)^\top \gamma^0\gamma^2 = -\gamma^\mu$$

(S.39)

in the Weyl basis, which in terms follows from $\gamma^1$ and $\gamma^3$ being anti-symmetric matrices commuting with the $\gamma^0\gamma^2$, while $\gamma^0$ and $\gamma^2$ are symmetric matrices anticommuting with the $\gamma^0\gamma^2$. 

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Altogether, we have

\[ \mathcal{C} : \mathcal{L} = \overline{\Psi}(i\slashed{\partial} - m)\Psi \mapsto \mathcal{L} + \text{a total derivative}, \quad (S.40) \]

and the Dirac action \( \int d^4x \mathcal{L} \) is \( \mathcal{C} \)-invariant.