Problem 1(a):
The ‘Dirac sandwich’
\[
\bar{v}\gamma_\nu u = v^\dagger \gamma^0 \gamma_\nu u = v^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} u
\]  
(S.1)
does not mix chiralities: if \( u \) and \( v \) are chiral, then they should have the same chirality, both left or both right, or else \( \bar{v}\gamma_\nu u = 0 \). By inspection of eqs. (2), the \( u \) spinor of an ultra-relativistic electron has chirality matching the electron’s helicity, left for \( \lambda = -\frac{1}{2} \) and right for \( \lambda = +\frac{1}{2} \). But the \( v \) spinor or an ultra-relativistic positron has the opposite chirality, left for \( \lambda = +\frac{1}{2} \) and right for \( \lambda = -\frac{1}{2} \). Thus, to have the same chiralities of \( u \) and \( v \) for the sake of \( \bar{v}\gamma_\nu u \), the electron and the positron must have opposite helicities. If the have the same helicities as in eq. (3), they have opposite chiralities and \( \bar{v}\gamma_\nu u = 0 \). \ Q.E.D. 

Indeed, let’s calculate the sandwich (S.1) for the spinors (2):

\[
\bar{v}(e^+_L)\gamma_\nu u(e^-_L) = -2E \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix} = 0,
\]  
(S.2)

\[
\bar{v}(e^+_L)\gamma_\nu u(e^-_R) = -2E \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix} = -2E \times \eta_L^\dagger \bar{\sigma}_\nu \xi_R,
\]  
(S.3)

\[
\bar{v}(e^+_R)\gamma_\nu u(e^-_L) = +2E \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix} = +2E \times \eta_R^\dagger \sigma_\nu \xi_L,
\]  
(S.3)

\[
\bar{v}(e^+_R)\gamma_\nu u(e^-_R) = +2E \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix} = 0.
\]  
(S.3)

Eqs. (3) have important practical consequences for electron-positron colliders. Any kind of fermion pair production — \( \mu^-\mu^+ \), or \( \tau^-\tau^+ \), or \( q\bar{q} \) — which proceeds through a virtual vector particle — a photon, or \( Z^0 \), or even something not yet discovered — would have the \( \bar{v}(e^+_\nu)\gamma_\nu u(e^-) \) factor in the amplitude. According to eq. (3), the electron and the positron must have opposite helicities, or they would not annihilate each other and make pairs.

Now suppose we have a longitudinally polarized electron beam — say \( \lambda = +\frac{1}{2} \) only — but the positron beam is un-polarized. Because of eq. (3), only the left-handed positrons would
collide with the right-handed electrons and produce pairs, while the left-handed positrons would do something else. Likewise, if we give the electron beam \( \lambda = -\frac{1}{2} \) polarization, then only the right-handed positrons would collide with our left-handed electrons and make pairs, while the left-handed positrons would do something else. Thus, as far as the pair-production is concerned, the positron beam could just as well be longitudinally polarized with \( \lambda(e^+) = -\lambda(e^-) \).

In other words, if we want to study polarization effects in fermion pair production, it’s enough to longitudinally polarize just the electron beam. We do not need to polarize the positron beam — which is much harder to do — because electrons of definite helicity would automatically select positrons of the opposite helicity.

**Problem 1(b):**
For ultra-relativistic muons, the \( u(\mu^-) \) and \( v(\mu^+) \) are chiral, and the chiralities behave exactly similar to the electron and positron in part (a): the Dirac sandwich \( \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \) vanishes unless \( u \) and \( v \) have the same chirality and hence the \( \mu^- \) and the \( \mu^+ \) have opposite helicities. Indeed,

\[
\bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) = -2E\begin{pmatrix} \xi_L \\eta_L \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix} = 0, \tag{4}
\]

\[
\bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) = -2E\begin{pmatrix} \xi_L \\eta_L \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} \eta_R \end{pmatrix} = +2E \times \xi_L^\dagger \sigma_\nu \eta_R, \tag{S.4}
\]

\[
\bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) = -2E\begin{pmatrix} 0 \\eta_L \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix} = -2E \times \xi_R^\dagger \sigma_\nu \eta_L, \tag{S.5}
\]

\[
\bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) = -2E\begin{pmatrix} 0 \\eta_L \end{pmatrix}^\dagger \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \bar{\sigma}_\nu \end{pmatrix} \begin{pmatrix} \eta_R \end{pmatrix} = 0. \tag{4}
\]

Eqs. (4) — and similar formulae for other fermion-antifermion pairs produced with ultra-relativistic speeds in electron-positron collisions — assure that the fermion and the antifermion always have opposite helicities. Experimentally, this means that if we somehow manage to determine the fermion’s helicity in some event, the for that even we also know the antifermion’s helicity and vice verse.
Problem 1(c):
The electron moves in the positive $z$ direction, so its helicity $\lambda$ is the same as its $S_z$ — the $z$ component of its spin. Hence, the $\xi$ spinors corresponding to the 2 helicities are

$$\xi(e^L_L) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \xi(e^L_R) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (S.6)$$

The positron moves in the negative $z$ direction, so its helicity is opposite from $S_z$, hence

$$\xi(e^+_L) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi(e^+_R) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (S.7)$$

The $\eta$ spinors in eqs. (2) for the positrons are related to $\xi$ spinors as $\eta = \sigma_2 \xi$, thus

$$\eta(e^+_L) = \begin{pmatrix} 0 \\ +i \end{pmatrix} \quad \text{and} \quad \eta(e^+_R) = \begin{pmatrix} -i \\ 0 \end{pmatrix}. \quad (S.8)$$

Substituting these 2–component spinors into eqs. (S.2) and (S.3), we obtain

$$\bar{v}(e^+_L)\gamma^\nu u(e^-_R) = -2E \times (0 \ \ -i) \bar{\sigma}^\nu \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= +2iE \times (\bar{\sigma}^\nu)_{21}$$

$$= 2E \times (0, +i, +1, 0)^\nu, \quad (5)$$

$$\bar{v}(e^+_R)\gamma^\nu u(e^-_L) = +2E \times (+i \ \ 0) \sigma^\nu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= +2iE \times (\sigma^\nu)_{12}$$

$$= 2E \times (0, -i, +1, 0)^\nu.$$
Problem 1(d):
Suppose for a moment $\theta = 0$ and the $\mu^+$ move in the same directions as $e^\mp$. Then the muons have exactly the same spinors $u$ and $v$ as the $e^\mp$ of the same charge and helicity, hence
\[
\bar{v}(\mu^+_L)\gamma^\nu u(\mu^-_R) = 2E \times (0, +i, +1, 0)\nu \quad \text{and} \quad \bar{v}(\mu^+_R)\gamma^\nu u(\mu^-_L) = 2E \times (0, -i, +1, 0)\nu \quad (S.9)
\]
which exactly as in eqs. (5). The $\bar{u}(\mu^-)\gamma^\nu v(\mu^+)$ Dirac sandwiches we need for the amplitude (1) follow by complex conjugation:
\[
(\bar{v}\gamma^\nu u) = \bar{u}\gamma^\nu v = \bar{u}\gamma^\nu v,
\]
and hence
\[
\bar{u}(\mu^-_R)\gamma^\nu v(\mu^+_L) = 2E \times (0, -i, +1, 0)\nu, \\
\bar{u}(\mu^-_L)\gamma^\nu v(\mu^+_R) = 2E \times (0, +i, +1, 0)\nu \quad (S.11)
\]
Eqs. (S.11) apply for $\theta = 0$. For other muon directions, we may simply rotate the 4–vectors (S.11) through angle $\theta$ in the $xz$ plane, thus
\[
\bar{u}(\mu^-_R)\gamma^\nu v(\mu^+_L) = 2E \times (0, -i \cos \theta, +1, +i \sin \theta)\nu, \\
\bar{u}(\mu^-_L)\gamma^\nu v(\mu^+_R) = 2E \times (0, +i \cos \theta, +1, -i \sin \theta)\nu \quad (7)
\]
Problem 1(e):
Substituting the Dirac sandwiches (5) and (7) into the pair production amplitude (1), we obtain
\[
\langle \mu^-_L, \mu^+_R \mid \mathcal{M} \mid e^-_L, e^+_R \rangle = \langle \mu^-_R, \mu^+_L \mid \mathcal{M} \mid e^-_R, e^+_L \rangle = -e^2 \times (1 + \cos \theta), \\
\langle \mu^-_R, \mu^+_L \mid \mathcal{M} \mid e^-_L, e^+_R \rangle = \langle \mu^-_L, \mu^+_R \mid \mathcal{M} \mid e^-_R, e^+_L \rangle = -e^2 \times (1 - \cos \theta) \quad (S.12)
\]
while all the other polarized amplitudes vanish by eqs. (3) and (4):
\[
\langle \mu^-_{\text{any}}, \mu^+_{\text{any}} \mid \mathcal{M} \mid e^-_L, e^+_R \rangle = \langle \mu^-_{\text{any}}, \mu^+_{\text{any}} \mid \mathcal{M} \mid e^-_R, e^+_L \rangle = 0, \\
\langle \mu^-_L, \mu^+_L \mid \mathcal{M} \mid e^-_{\text{any}}, e^+_{\text{any}} \rangle = \langle \mu^-_R, \mu^+_R \mid \mathcal{M} \mid e^-_{\text{any}}, e^+_{\text{any}} \rangle = 0 \quad (S.13)
\]
The partial cross-sections (8) follow from these amplitudes according to
\[
\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \times \left( \frac{|\mathbf{p}'|}{|\mathbf{p}|} = 1 \right). \quad (S.14)
\]
Problem 1(f):
Summing the polarized cross-sections (8) over the muons’ helicities, we get
\[
\frac{d\sigma(e_L^- + e_R^+ \to \mu^-_{\text{any}} + \mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} = \frac{d\sigma(e_R^- + e_L^+ \to \mu^-_{\text{any}} + \mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1 + \cos^2 \theta)^2 + \frac{\alpha^2}{4s} \times (1 - \cos^2 \theta)^2 + 0 + 0
\]
while
\[
\frac{d\sigma(e_L^- + e_R^+ \to \mu^-_{\text{any}} + \mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} = \frac{d\sigma(e_R^- + e_L^+ \to \mu^-_{\text{any}} + \mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} = 0. \tag{S.16}
\]
Averaging these cross-sections over the electron’s and positron’s helicities gives
\[
\frac{d\sigma(e_{\text{avg}}^- + e_{\text{avg}}^+ \to \mu^-_{\text{any}} + \mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} = \frac{1}{4} \left( \frac{\alpha^2}{2s} \times (1 + \cos^2 \theta) + \frac{\alpha^2}{2s} \times (1 + \cos^2 \theta) + 0 + 0 \right)
\]
\[
= \frac{\alpha^2}{4s} \times (1 + \cos^2 \theta), \tag{S.17}
\]
which is exactly what we found in class for the un-polarized cross-section for \(E \gg M_\mu\).

Problem 2(a):
The amplitude for the diagram (10) comprises 3 factors:
- \(\bar{u}(p', s')(+ie\gamma^\nu)u(p, s)\) for the electrons’ external legs and the electron-photon vertex;
- \(-iZe \times 2M_N \delta^{\mu0}\) for the nuclear external legs and the nucleus-photon vertex, \textit{cf.} eq. (9);
- \(-ig_{\mu\nu}/q^2\) for the photon propagator.

Putting all the factors together, we obtain
\[
i\mathcal{M} = \bar{u}(p', s')(+ie\gamma^\nu)u(p, s) \times (-2iM_N Ze)\delta^{\mu0} \times \frac{-ig_{\mu\nu}}{q^2} = \frac{-2iM_N Ze^2}{q^2} \times \bar{u}(p', s')\gamma^0 u(p, s). \tag{S.18}
\]
Problem 2(b):
For an elastic scattering, the polarized partial cross-section is

\[
\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{|\mathcal{M}|^2}{64\pi^2 s}
\]

where \(\sqrt{s}\) is the total energy of the process in the CM frame. As long as the electron’s energy and momentum are much smaller than the nucleus’s mass \(M_N\), we may neglect the nucleus’s recoil and approximate \(\sqrt{s} = E_N + E_e \approx M_N + E_e \approx M_N\). Also, we may neglect the difference between the center-of-mass frame and the lab frame (where the nucleus is initially at rest). Thus,

\[
\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 M_N^2} = \frac{Z^2\alpha^2}{(q^2 = -q^2)^2} \times \left| \bar{u}(p', s')\gamma^0 u(p, s) \right|^2.
\]

When the initial electron beam is un-polarized and the final electron’s spin isn’t detected, we should average this cross-section over \(s\) and sum over the \(s'\), hence

\[
\overline{\frac{d\sigma}{d\Omega}} = \frac{Z^2\alpha^2}{(q^2)^2} \times \frac{1}{2} \sum_s \sum_{s'} \left| \bar{u}(p', s')\gamma^0 u(p, s) \right|^2.
\]

Problem 2(c):
As explained in class,

\[
\sum_{s,s'} \left| \bar{u}(p', s')\gamma^0 u(p, s) \right|^2 = \sum_{s,s'} \bar{u}(p', s')\gamma^0 u(p, s) \times \bar{u}(p, s)\gamma^0 u(p', s')
\]

\[
= \text{tr} \left( (p' + m_e)\gamma^0(p + m_e)\gamma^0 \right)
\]

\[
= \text{tr}(p'\gamma^0\gamma^0) + m_e \text{tr}(\gamma^0\gamma^0) + m_e \text{tr}(p'\gamma^0\gamma^0) + m_e^2 \text{tr}(\gamma^0\gamma^0)
\]

\[
= 4(p^0)(p^0) + 4(p'^0)(p^0) - 4(p'p)g^{00}
\]

\[
+ m \times 0 + m \times 0 + m_e^2 \times 4g^{00}
\]

\[
= 2 \times 4E'E - 4(p'p) + 4m_e^2
\]

and consequently

\[
\frac{1}{2} \sum_{s,s'} \left| \bar{u}(p', s')\gamma^0 u(p, s) \right|^2 = 2 \left( m_e^2 + 2EE' - (pp' = EE' - p \cdot p') \right) = 2(m_e^2 + EE' + p \cdot p').
\]
Problem 2(d):
Mott scattering is elastic, so $E' = E$, $|p'| = |p|$, hence $p \cdot p' = p^2 \cos \theta$, and in eq. (12)

$$m_e^2 + EE' + p \cdot p' = m_e^2 + E^2 + p^2 \times (\cos \theta = 1 - 2 \sin^2(\theta/2)) = 2E^2 - 2p^2 \sin^2(\theta/2).$$  

(S.22)

Also,

$$q^2 = (p' - p)^2 = p^2 + p'^2 - 2p \cdot p' = p^2 \times (2 - 2 \cos \theta) = 4p^2 \times \sin^2(\theta/2).$$  

(S.23)

Substituting these formulae into eq. (11), we obtain

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} = \frac{(Z \alpha)^2}{(4p^2 \sin^2(\theta/2))^2} \times 4(E^2 - p^2 \sin^2(\theta/2)).$$  

(S.24)

Now let’s compare this formula to the Rutherford cross-section for the non-relativistic electrons,

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{Rutherford}} = \frac{(Ze^2)^2}{4m_e^2 v_e^4 \sin^4(\theta/2)}$$  

in Gaussian units, or eq. (14) in the rationalized $c = \hbar = 1$ units. Incidentally, the Rutherford formula is true in both classical mechanics and quantum mechanics, and in QM it is given by the first Born approximation but is also exact. Comparing Mott scattering (S.24) to the Rutherford scattering, we see that

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Rutherford}} \times \frac{m_e \beta^4}{P^4} \times (E^2 - p^2 \sin^2(\theta/2))$$

$$= \left( \frac{d\sigma}{d\Omega} \right)_{\text{Rutherford}} \times \frac{1 - \beta^2 \sin^2(\theta/2)}{\gamma^2}.$$  

(13)

Quod erat demonstrandum.
Problem 3(a):
First, a point of notations: the initial electron has momentum $p_1$, the final electron’s momentum is $p_1'$, while the initial and final positrons have respective momenta $p_2$ and $p_2'$. The Lorenz invariant products $s, t, u$ of these momenta are as in eq. (18).

The first diagram (15) gives us

$$iM_1 = -\left( \bar{v}(e^+)(ie\gamma_\mu)v(e^+) \right) \times \left( \bar{u}(e^-)(ie\gamma_\nu)u(e^-) \right) \times \frac{-ig^{\mu\nu}}{q^2}$$

(S.26)

where the overall minus sign is due to the positron-out to positron-in fermionic line. Also, $q = p_1' - p_1 = p_2 - p_2'$ and hence $q^2 = t$, cf. the second eq. (18).

For the second diagram (15) we have

$$iM_2 = +\left( \bar{v}(e^+)(ie\gamma_\mu)u(e^-) \right) \times \left( \bar{u}(e^-)(ie\gamma_\nu)v(e^+) \right) \times \frac{-ig^{\mu\nu}}{q^2}$$

(S.27)

where the overall sign is plus because both fermionic lines have an incoming or outgoing electron at one end. Also, $\tilde{q} = p_1 + p_2 = p_1' + p_2'$ and hence $\tilde{q}^2 = s$, cf. the first eq. (18).

Problem 3(b):
Summing /averaging the $|M_2|^2$ over spins works exactly as for the muon pair production discussed in class:

$$\sum_{\text{spins}} |M_2|^2 = \left( \frac{e^2}{s} \right)^2 \sum_{\text{spins}} \left[ \bar{v}(e^+)( ie\gamma_\mu )u(e^-) \times \bar{u}(e^-)(ie\gamma_\nu)v(e^+) \right] \times \left[ \bar{u}(e^-)(ie\gamma_\mu)v(t+) \times \bar{v}(e^-)(ie\gamma_\nu)u(t-) \right]$$

$$= \left( \frac{e^2}{s} \right)^2 \text{tr} \left[ (p_2 - m) \gamma_\mu(p_1 + m)\gamma_\nu \right] \times \text{tr} \left[ (p_1' - m) \gamma_\mu(p_2' - m)\gamma_\nu \right]$$

$$\approx \left( \frac{e^2}{s} \right)^2 \text{tr} \left[ p_2 \gamma_\mu p_1 \gamma_\nu \right] \times \text{tr} \left[ p_1' \gamma_\mu p_2' \gamma_\nu \right]$$

(S.28)

$$= \left( \frac{e^2}{s} \right)^2 \times 4 \left[ p_{2\mu}p_{1\nu} + p_{2\nu}p_{1\mu} - g_{\mu\nu}(p_2p_1) \right] \times 4 \left[ p_{2'\mu}p_{1'\nu} + p_{2'\nu}p_{1'\mu} - g_{\mu\nu}(p_2'p_1') \right]$$

8
\[= 16 \left(\frac{e^2}{s}\right)^2 \left[2(p_2'p_2)(p_1'p_1) + 2(p_2'p_1)(p_1'p_2) \right.\]
\[\left. - 2(p_2'p_1)(p_2p_1) - 2(p_2'p_1')(p_2p_1) - 42(p_2'p_1')(p_2p_1)\right] \]
\[= 32 \left(\frac{e^2}{s}\right)^2 \left[(p_2'p_2)(p_1') + (p_2'p_1)(p_1'p_2)\right] \]
\[= 8 \left(\frac{e^2}{s}\right)^2 [t^2 + u^2]\]

(S.28)

where the last equality follows from the kinematic relations (18). Altogether,

\[
\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_2|^2 = 2e^4 \times \frac{t^2 + u^2}{s^2}.
\]

Problem 3(c):
The two diagrams for Bhabha scattering are related by the *crossing symmetry*, so the amplitudes \(\mathcal{M}_1\) and \(\mathcal{M}_2\) are related to each other via analytic continuation of particle’s momenta. In terms of the spin-summed \(|\mathcal{M}|^2\) and Mandelstam variables (18),

\[
\sum_{\text{spins}} |\mathcal{M}_1(s, t, u)|^2 = \sum_{\text{spins}} |\mathcal{M}_2(t, s, u)|^2,
\]

(S.29)

hence eq. (19) for the second amplitude implies a similar equation for the first amplitude, but with \(s\) and \(t\) exchanged with each other — *i.e.*, eq. (20).

Alternatively, we may sum the \(|\mathcal{M}_1|^2\) over all the spins in the same way as we summed the \(|\mathcal{M}_2|^2\) in part (b):

\[
\sum_{\text{spins}} |\mathcal{M}_1|^2 = \left(\frac{e^2}{t}\right)^2 \sum_{\text{spins}} \left[\bar{u}(e^-)\gamma^\mu u(e^-) \times \bar{u}(e^-)\gamma^\nu u(e^-)\right] \times \left[\bar{v}(e^+)\gamma_\mu v(e^+) \times \bar{v}(e^+)\gamma_\nu v(e^+)\right]
\]
\[= \left(\frac{e^2}{t}\right)^2 \text{tr} \left[(p_1' + m)\gamma^\mu (p_1 + m)\gamma^\nu\right] \times \text{tr} \left[(p_2' - m)\gamma_\mu (p_2' - m)\gamma_\nu\right]
\]
\[\approx \left(\frac{e^2}{t}\right)^2 \text{tr} \left[p_2'\gamma^\mu p_1\gamma^\nu\right] \times \text{tr} \left[p_2\gamma_\mu p_1\gamma_\nu\right]
\]
\[= \left(\frac{e^2}{t}\right)^2 \times 4 \left[p_1'p_1 + p_1p_1' - g^\mu\nu(p_1p_1')\right] \times 4 \left[p_2'p_2 + p_2p_2' - g_{\mu\nu}(p_2p_2')\right]
\]

(S.30)
\[ = 16 \left( \frac{e^2}{t} \right)^2 \left[ 2(p_1^2 p_2^2)(p_1 p_2) + 2(p_1^2 p_2)(p_1 p_2') \\
- 2(p_1^2 p_1)(p_2 p_2') - 2(p_1^2 p_1)(p_2' p_2) + 4(p_1^2 p_1)(p_2 p_2') \right] \]
\[ = 32 \left( \frac{e^2}{t} \right)^2 \left[ (p_1' p_2')(p_1 p_2) + (p_1' p_2)(p_1 p_2') \right] \]
\[ = 8 \left( \frac{e^2}{t} \right)^2 \left[ s^2 + u^2 \right] \] (S.30)

and hence
\[ \frac{1}{4} \sum_{\text{spins}} |M_1|^2 = 2e^4 \times \frac{s^2 + u^2}{t^2}. \] (20)

**Problem 3(d):**

The interference term between the two diagrams is more complicated:

\[ M_1^* \times M_2 = -\frac{e^2}{t} \left( \bar{u}(e^-) \gamma^\nu u(e^-') \times \bar{v}(e^+) \gamma_\mu v(e^+) \right) \times \]
\[ \times \frac{e^2}{s} \left( \bar{v}(e^+) \gamma_\mu u(e^-) \times \bar{u}(e^-') \gamma^\nu v(t+) \right) \]
\[ = -\frac{e^4}{st} \times \bar{u}(e^-) \gamma^\nu u(e^-') \times \bar{u}(e^-') \gamma^\mu v(t+) \times \bar{v}(e^+) \gamma_\mu v(e^+) \times \bar{v}(e^+) \gamma_\mu u(e^-), \] (S.31)

so summing over all the spins produces a single big trace rather than a product of all traces,

\[ \sum_{\text{spins}} M_1^* \times M_2 = -\frac{e^4}{st} \times \text{tr} \left[ (\not{p}_1 + m) \gamma^\nu (\not{p}_1' + m) \gamma_\mu (\not{p}_2 - m) \gamma_\nu (\not{p}_2 - m) \gamma_\mu \right] \] (S.32)
\[ \approx -\frac{e^4}{st} \times \text{tr} \left[ \not{p}_1 \gamma^\nu \not{p}_1' \gamma_\mu \not{p}_2 \gamma_\nu \not{p}_2 \gamma_\mu \right]. \]

This trace looks more complicated than it is, and we may drastically simplify it by summing over \( \nu \) and \( \mu \) before taking the trace. Back in homework set #8 (problem 2(d)) we saw that

\[ \gamma^\alpha \not{d} \not{y} \not{y} \gamma_\alpha = -2 \not{y} \not{d} \not{y} \quad \text{and} \quad \gamma^\alpha \not{d} \not{y} \gamma_\alpha = 4(ab). \] (S.33)
For the problem at hand, this gives us $\gamma^\nu p_1^\nu \gamma^\mu p_2^\mu = -2 p_2^\nu \gamma^\mu p_1^\mu$ and hence

\[
\text{tr} \left[ p_1 p_2 \gamma^\nu p_2^\nu \gamma^\mu p_2^\mu\right] = -2 \text{tr} \left[ p_1 p_2^\nu \times 4 (p_1 p_2^\mu) \right] = -8(p_1 p_2) \times \text{tr} \left[ p_1 p_2^\mu\right]
\]

Consequently,

\[
\frac{1}{4} \sum_{\text{spins}} M_1^* \times M_2 = +2e^4 \times \frac{u^2}{st}.
\]

**Problem 3(e):**

Assembling spin sums / averages (19–21) together according to eq. (17), we get

\[
|M|^2 \overset{\text{def}}{=} \frac{1}{4} \sum_{\text{spins}} |M_1 + M_2|^2
\]

\[
= \frac{1}{4} \sum_{\text{spins}} \left( |M_1|^2 + |M_2|^2 + 2\text{Re} M_1^* M_2 \right)
\]

\[
= 2e^4 \times \frac{s^2 + u^2}{t^2} + 2e^4 \times \frac{t^2 + u^2}{s^2} + 4e^4 \times \frac{u^2}{st} \quad (S.35)
\]

Consequently, the un-polarized partial cross-section for the Bhabha scattering is

\[
\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{|M|^2}{64\pi^2 s} = \frac{\alpha^2}{2s} \times \frac{s^4 + t^4 + u^4}{s^2 \times t^2}.
\]

To complete the problem, let’s do the kinematics. In the center of mass frame

\[
s = 4E^2 \approx 4p^2,
\]

\[
t = -(p_1' - p_1)^2 = -2p^2(1 - \cos \theta),
\]

\[
u = -(p_2' - p_1)^2 = -2p^2(1 + \cos \theta),
\]

\[
(S.37)
\]
hence

\[
\frac{s^4 + t^4 + u^4}{s^2 t^2} = \frac{(2p^2)^4 \times (2^4 + (1 - \cos \theta)^4 + (1 + \cos \theta)^4)}{(2p^2)^4 \times 2^2 \times (1 - \cos \theta)^2} = \frac{(3 + \cos^2 \theta)^2}{2(1 - \cos \theta)^2}.
\] (S.38)

Plugging this formula into eq. (S.36) finally gives us

\[
\frac{d\sigma^{\text{Bhabha}}}{d\Omega_{\text{cm}}} = \frac{\alpha^2}{4s} \times \frac{(3 + \cos^2 \theta)^2}{(1 - \cos \theta)^2}.
\] (S.39)

*Quod erat demonstrandum.*