The one-loop diagram (1) yields amplitude
\[
F(\delta k) = \frac{-i\lambda^2}{2} \int \frac{d^4p_1}{(2\pi)^4} \frac{1}{p_1^2 - m^2 + i0} \times \frac{1}{(p_2 = \delta k - p_1)^2 - m^2 + i0}, \tag{S.1}
\]
but the momentum integral here diverges logarithmically as \( p_1 \to \infty \), so it needs to be regularized. In the Pauli–Villars regularization scheme, one subtracts from (1) a similar diagram where internal lines belong to ghost fields of extremely large mass \( \Lambda \gg m \). The subtraction is done before the momentum integration,
\[
F_{PV}(\delta k) = \frac{-i\lambda^2}{2} \int \frac{d^4p_1}{(2\pi)^4} \left\{ \frac{1}{p_1^2 - m^2 + i0} \times \frac{1}{(p_2 = \delta k - p_1)^2 - m^2 + i0} \right. \\
- \frac{1}{p_1^2 - \Lambda^2 + i0} \times \frac{1}{(p_2 = \delta k - p_1)^2 - \Lambda^2 + i0} \right\}, \tag{S.2}
\]
so for \( p^2 \gtrsim \Lambda^2 \) the net integrand behaves as \( O(\Lambda^2/p^6) \) instead of \( 1/p^4 \) and the integral converges.

Our task is to evaluate the integral (S.2), so let’s start with the Feynman’s trick for simplifying the propagator product. As discussed in class,
\[
\frac{1}{p_1^2 - m^2 + i0} \times \frac{1}{(p_2 = \delta k - p_1)^2 - m^2 + i0} = \\
= \int_0^1 dx \left\{ \frac{1}{[(1-x)(p_1^2 - m^2 + i0) + x(p_2^2 - m^2 + i0)]^2} \right. \\
\left. = \int_0^1 dx \frac{1}{[q^2 + x(1-x) \times \delta k^2 - m^2 + i0]^2} \tag{S.3}\right.
\]
where \( q = p_1 - x \times \delta k \).

Similarly,
\[
\frac{1}{p_1^2 - \Lambda^2 + i0} \times \frac{1}{(p_2 = \delta k - p_1)^2 - \Lambda^2 + i0} = \int_0^1 dx \frac{1}{[q^2 + x(1-x) \times \delta k^2 - \Lambda^2 + i0]^2} \tag{S.4}
\]
for exactly same \( q = p_1 - x\delta k \). Hence, we plug both propagator products into eq. (S.2),
change the order of integration, and then change the momentum variable from $p$ to $q$,

$$F_{PV}(\delta k^2 = t) = \int_0^1 dx F_{PV}(t, x) \quad \text{(S.5)}$$

where

$$F_{PV}(t, x) = \frac{-i\lambda^2}{2} \int \frac{d^4 p_1}{(2\pi)^4} \left\{ \frac{1}{[(q = p_1 - x\delta k)^2 + tx(1 - x) - m^2 + i0]^2} - \frac{1}{[(q = p_1 - x\delta k)^2 + tx(1 - x) - \Lambda^2 + i0]^2} \right\}$$

$$= \frac{-i\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} \left\{ \frac{1}{[q^2 + tx(1 - x) - m^2 + i0]^2} - \frac{1}{[q^2 + tx(1 - x) - \Lambda^2 + i0]^2} \right\}. \quad \text{(S.6)}$$

Next, we analytically continue the momentum integral from the Minkowski momentum $q^\mu$ to the Euclidean Momentum $q_E^\mu$, thus

$$d^4 q \rightarrow id^4 q_E, \quad q^2 \rightarrow -q_E^2, \quad \text{(S.7)}$$

and hence

$$F_{PV}(t, x) = \frac{\lambda^2}{2} \int \frac{d^4 q_E}{(2\pi)^4} \left\{ \frac{1}{[q_E^2 + m^2 - x(1 - x)t]^2} - \frac{1}{[q_E^2 + \Lambda^2 - x(1 - x)t]^2} \right\}. \quad \text{(S.8)}$$

At this point, we go to spherical coordinates in the 4D Euclidean space and focus on the radial coordinate $|q_E|$. This gives us

$$d^4 q_E = 2\pi^2 |q_E|^3 dq_E = \pi^2 q_E^2 dq_E \quad \text{(S.9)}$$

and hence

$$F_{PV}(t, x) = \frac{\lambda^2}{32\pi^2} \int_0^\infty dq_E^2 \left\{ \frac{q_E^2}{[q_E^2 + m^2 - x(1 - x)t]^2} - \frac{q_E^2}{[q_E^2 + \Lambda^2 - x(1 - x)t]^2} \right\}. \quad \text{(S.10)}$$
The last integral here has form

\[ \int_0^\infty dy \left( \frac{1}{(y + A)^2} - \frac{1}{(y + B)^2} \right) \]  

which evaluates to \( \log(B/A) \). Indeed,

\[ \int_0^\infty dy \left( \frac{y}{(y + A)^2} - \frac{y}{(y + B)^2} \right) = \int_0^\infty dy \left( \frac{1}{y + A} - \frac{1}{y + B} - \frac{A}{(y + A)^2} + \frac{B}{(y + B)^2} \right) \]

\[ = \left( \log \frac{y + A}{y + B} + \frac{A}{y + A} - \frac{B}{y + B} \right) \bigg|_0^\infty \]

\[ = \left( \log \frac{A + \infty}{B + \infty} - \log \frac{A}{B} \right) + \left( \frac{A}{\infty} - \frac{A}{A} \right) - \left( \frac{B}{\infty} - \frac{B}{B} \right) \]

\[ = \left( 0 - \log \frac{A}{B} \right) + (0 - 1) - (0 - 1) \]

\[ = \log \frac{B}{A} \].  

Hence,

\[ F_{PV}(t, x) = \frac{\lambda^2}{32\pi^2} \times \log \frac{\Lambda^2 - x(1 - x)t}{m^2 - x(1 - x)t} \approx \frac{\lambda^2}{32\pi^2} \times \log \frac{\Lambda^2}{m^2 - x(1 - x)t} \]

since we assume not only \( \Lambda \gg m \) but also \( \Lambda^2 \gg |t|, |u|, |s| \).

Integrating this formula over \( x \), we arrive at the Pauli–Villars regularized amplitude,

\[ F_{PV}(t) = \frac{\lambda^2}{32\pi^2} \int_0^1 dx \log \frac{\Lambda^2}{m^2 - x(1 - x)t} \approx \frac{\lambda^2}{32\pi^2} \left( \log \frac{\Lambda^2}{m^2} + I \left( \frac{t}{m^2} \right) \right) \]  

where

\[ I \left( \frac{t}{m^2} \right) \overset{\text{def}}{=} \int_0^1 dx \log \frac{m^2}{m^2 - x(1 - x)t}. \]  

In class, we have calculated the same one-loop amplitude using the hard-edge cutoff; our
result was

\[ F_{\text{hard edge}} = \frac{\lambda^2}{32\pi^2} \left( \log \frac{\Lambda^2}{m^2} - 1 + I \left( \frac{t}{m^2} \right) \right) \]  

(S.16)

where \( I(t/m^2) \) is exactly as in eq. (S.15). Comparing eqs. (S.16) and (S.14), we immediately see that the only difference is the \(-1\) term inside the parentheses in eq. (S.16). In other words, the two regularization schemes produce similar amplitudes except for a \textit{constant} term \((\lambda^2/32\pi^2) \times O(1)\). Moreover, this constant term may be eliminated by adjusting the cutoff scales of the two regulators.

Indeed, the cutoff scale \( \Lambda_{\text{HE}} \) of the hard-edge regulator — the maximal value of the Euclidean momenta allowed in that scheme — does not have to be exactly the same as the mass \( \Lambda_{\text{PV}} \) of the ghost fields in the Pauli–Villars regularization scheme. To produce a similar physical effect, the two scales should have similar orders of magnitude, but this generally means

\[ \Lambda_{\text{HE}}^2 = \Lambda_{\text{PV}}^2 \times c \]  

(2)

for some \( O(1) \) constant \( c \) rather than naive identification \( \Lambda_{\text{HE}} = \Lambda_{\text{PV}} \). Consequently, in the Pauli–Villars regularization scheme

\[ F(t) = \frac{\lambda^2}{32\pi^2} \left( \log \frac{\Lambda_{\text{PV}}^2}{m^2} + I \left( \frac{t}{m^2} \right) \right) \]  

(S.17)

while in the hard-edge regularization scheme

\[ F(t) = \frac{\lambda^2}{32\pi^2} \left( \log \frac{\Lambda_{\text{HE}}^2}{m^2} - 1 + I \left( \frac{t}{m^2} \right) \right), \]  

(S.18)

and \( \log \Lambda_{\text{PV}}^2 \) may be different from \( \log \Lambda_{\text{HE}}^2 \) by some \( O(1) \) constant \( \log c \) according to eq. (2).

Thus, \textit{the one-loop amplitudes} (S.17) and (S.18) are in \textit{perfect agreement with each other}, provided we identify

\[ \log \Lambda_{\text{PV}}^2 = \Lambda_{\text{HE}}^2 - 1, \]  

(S.19)

\textit{i.e.,} \( \Lambda_{\text{PV}}^2 = \Lambda_{\text{HE}}^2 \times \exp(-1) \).
Now consider the higher-derivative regularization scheme. In this scheme, the scalar field \( \phi \) has very small higher-derivative terms in its Lagrangian,

\[
\mathcal{L}_{\text{HD}} = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{24} \phi^4 - \frac{1}{2\Lambda^2} (\partial^2 \phi)^2,
\]

which softens the scalar’s propagator for very high momenta \( p^2 \gtrsim \Lambda^2 \):

\[
\frac{i}{p^2 - m^2 + i0} \rightarrow \frac{i}{p^2 - m^2 + i0 - \Lambda^{-2} p^4} \approx \frac{i}{p^2 - m^2 + i0} \times \frac{-\Lambda^2}{p^2 - \Lambda^2 + i0}.
\]

Consequently, in the higher-derivative regularization scheme, the one-loop amplitude (1) becomes

\[
F_{\text{HD}}(\delta k) = \int_0^1 dx F_{\text{HD}}(\delta k, x) \]

where \( p_2 \equiv \delta k - p_1 \). For all but extremely large momenta \( p^2 \ll \Lambda^2 \), the integrand here is indistinguishable from the un-regularized loop integral (S.1), but for \( p^2 \gtrsim \Lambda^2 \) it becomes softer — behaves like \( \Lambda^4/p^8 \) for \( p^2 \rightarrow \infty \) instead of \( 1/p^2 \) — so the integral (S.22) converges.

Our task is to evaluate this integral, so let’s start by simplifying the propagator product by using the Feynman’s trick (S.3) and then interchanging the order of \( dx \) and \( d^4p_1 \) integrals:

\[
F_{\text{HD}}(\delta k) = \int_0^1 dx F_{\text{HD}}(\delta k, x)
\]

where

\[
F_{\text{HD}}(\delta k, x) = \frac{-i\lambda^2}{2} \int \frac{d^4p_1}{(2\pi)^4} \frac{1}{p_1^2 - m^2 + i0 - \Lambda^{-2} p_1^4} \times \frac{-\Lambda^2}{p_2^2 - m^2 + i0} \times \frac{-\Lambda^2}{p_2^2 - \Lambda^2 + i0}.
\]

Note that we have used the Feynman trick only for the \( 1/(p_1^2 - m^2 + i0) \) and \( 1/(p_2^2 - m^2 + i0) \) factors, the remaining \( \Lambda \)-dependent factors remain as they are on the second line of eq. (S.24).
Next, inside the $\int dx$ integral, we change the momentum integration variable from $p_1$ to $q = p_1 - x\delta k$, thus

$$F_{\text{HD}}(\delta k, x) = \frac{-i\lambda^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{\left[q^2 + x(1-x)\delta k^2 - m^2 + i0\right]^2} \times \frac{-\Lambda^2}{(p_1 = q + x\delta k)^2 - \Lambda^2 + i0} \times \frac{-\Lambda^2}{(-p_2 = q + (x-1)\delta k)^2 - \Lambda^2 + i0} \quad (S.25)$$

The UV cutoff scale $\Lambda$ must be much larger than the scalar’s mass $m$ and also than any component $\delta k^\mu$ of the net momentum transfer $\delta k$. Consequently, for any $q^\mu$ we may approximate

$$\frac{-\Lambda^2}{(q + O(\delta k))^2 - \Lambda^2 + i0} = \approx \frac{-\Lambda^2}{q^2 - \Lambda^2 + i0} \quad (S.26)$$

For $q \ll \Lambda$ this approximation works because the whole $(q + O(\delta k))^2$ term in the denominator is negligible compared to the $-\Lambda^2$ term, while for $q \sim \Lambda$ or large, $O(\delta k)$ correction to $q$ becomes negligible because $\delta k \ll q$. Applying this approximation to both $\Lambda$–dependent factors in eq. (S.25), we have

$$\frac{-\Lambda^2}{(p_1 = q + x\delta k)^2 - \Lambda^2 + i0} \times \frac{-\Lambda^2}{(-p_2 = q + (x-1)\delta k)^2 - \Lambda^2 + i0} \approx \frac{\Lambda^4}{[q^2 - \Lambda^2 + i0]^2} \quad (S.27)$$

and hence

$$F_{\text{HD}}(\delta k, x) = \frac{-i\lambda^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{\left[q^2 + x(1-x)\delta k^2 - m^2 + i0\right]^2} \times \frac{\Lambda^4}{[q^2 - \Lambda^2 + i0]^2} \quad (S.28)$$

At this point, we analytically continue the momentum integral from the Minkowski momentum $q^\mu$ to the Euclidean momentum $q_E^\mu$: $d^4q$ becomes $id^4q_E$, $q^2$ becomes $-q_E^2$, and the integral (S.28) becomes

$$F_{\text{HD}}(\delta k^2 = t, x) = \frac{\lambda^2}{2} \int \frac{d^4q_E}{(2\pi)^2} \frac{1}{[q_E^2 + m^2 - x(1-x)t]^2} \times \frac{\Lambda^4}{[q_E^2 + \Lambda^2]^2} \quad (S.29)$$

On the second line here, we have integrated over the directions of the $q_E^\mu$ in the 4D Euclidean
space. As to the remaining radial integral, it has form

$$\int_0^\infty dy \frac{yB^2}{(y+A)^2(y+B)^2}$$

(S.30)

where $A = m^2 - x(1 - x)t$ is much less than $B = \Lambda^2$. The simplest way to evaluate this integral is to split it at some point $C$ which is much bigger than $A$ but much smaller that $B$. Thus

$$\int_0^\infty dy \frac{yB^2}{(y+A)^2(y+B)^2} = \int_0^C dy \frac{yB^2}{(y+A)^2(y+B)^2} + \int_C^\infty dy \frac{yB^2}{(y+A)^2(y+B)^2}$$

(S.31)

where in the zero to $C$ integral $y \leq C \ll B$ allows us to approximate

$$\int_0^C dy \frac{yB^2}{(y+A)^2(y+B)^2} \approx \int_0^C dy \frac{y}{(y+A)^2} = \log \frac{A+C}{A} - \frac{C}{C+A} \approx \log \frac{C}{A} - 1,$$  

(S.32)

while in the $C$ to infinity integral $y \geq C \gg A$ makes for $(y+A)^2 \approx y^2$ in the denominator and hence

$$\int_C^\infty dy \frac{yB^2}{(y+A)^2(y+B)^2} \approx \int_C^\infty dy \frac{B^2}{y(y+B)^2}$$

$$= \int_C^\infty dy \left\{ \frac{1}{y} - \frac{1}{y+B} - \frac{B}{(y+B)^2} \right\}$$

$$= \log \frac{B+C}{C} - \frac{B}{B+C}$$

$$\approx \log \frac{C}{B} - 1.$$  

(S.33)

Altogether, for $A \ll C \ll B$,

$$\int_0^\infty dy \frac{yB^2}{(y+A)^2(y+B)^2} \approx \log \frac{C}{A} - 1 + \log \frac{B}{C} 1 = \log \frac{B}{A} - 2.$$  

(S.34)

note that $C$ drops out of net result; if it did not, our approximations would be inconsistent.
Alternatively, we may evaluate the integral (S.30) without using any approximations by expanding the integrand — which is a rational function of \( y \) — into its poles:

\[
\frac{yB^2}{(y + A)^2(y + B)^2} = \frac{\alpha}{(y + A)^2} + \frac{\beta}{(y + B)^2} + \frac{\gamma}{y + A} + \frac{\delta}{y + B}
\]  
(S.35)

for some constants \( \alpha, \beta, \gamma, \delta \). The values of \( \alpha \) and \( \beta \) follow by matching the coefficients of the double poles at \( y = -A \) and \( y = -B \) at both sides, thus

\[
\alpha = -\frac{AB^2}{(B - A)^2}, \quad \beta = -\frac{B^3}{(B - A)^2}
\]  
(S.36)

Subtracting the double poles from both sides of eq. (S.35) and matching the residues of the remaining single poles, we obtain

\[
\gamma = \delta = \frac{+B^2(B + A)}{(B - A)^3}
\]  
(S.37)

Consequently,

\[
\int_0^\infty dy \frac{yB^2}{(y + A)^2(y + B)^2} = \frac{B^2}{(B - A)^2} \times \int_0^\infty dy \left[ \frac{B + A}{B - A} \left( \frac{1}{y + A} - \frac{1}{y + B} \right) - \frac{A}{(y + A)^2} - \frac{B}{(y + B)^2} \right]
\]  
(S.38)

\[
= \frac{B^2}{(B - A)^2} \times \left[ \frac{B + A}{B - A} \times \log \frac{B}{A} - 1 - 1 \right] \quad \langle \text{for any } B > A > 0 \rangle
\]

\[
\approx \log \frac{B}{A} - 2 \quad \langle \text{for } B \gg A \rangle,
\]

in perfect agreement with eq. (S.34).

Plugging this formula into the momentum integral (S.29), we obtain

\[
\mathcal{F}_{\text{HD}}(t, x) = \frac{\lambda^2}{32\pi^2} \times \left( \log \frac{\Lambda^2}{m^2 - x(1 - x)t} - 2 \right)
\]  
(S.39)
and consequently

\[ F_{\text{HD}}(t) = \frac{\lambda^2}{32\pi^2} \int_0^1 dx \left( \log \frac{\Lambda^2}{m^2} - x(1-x)t - 2 \right) \]

\[ = \frac{\lambda^2}{32\pi^2} \left( \log \frac{\Lambda^2}{m^2} - 2 + I \left( \frac{t}{m^2} \right) \right) \]  \hspace{1cm} (S.40)

where \( I(t/m^2) \) is as in eq. (S.15).

Similar to the Pauli–Villars case, the \( \Lambda^2 \) parameter of the higher-derivative regularization scheme does not have to be exactly equal to the hard-edge cutoff \( \Lambda_{\text{HE}}^2 \) to produce the same physical effect. Instead, we expect

\[ \Lambda_{\text{hard edge}}^2 = \Lambda_{\text{higher-derivative}}^2 \times c' \]  \hspace{1cm} (2)

for some \( O(1) \) numerical constant \( c' \). Consequently, \( \log \Lambda^2 \) in the amplitude (S.40) can be shifted by a constant, and in this way the one-loop diagram (1) regularized using the higher-derivative term become consistent with the other regulators — hard-edge and Pauli–Villars for the same amplitude. Indeed, eq. (S.40) re-written as

\[ F(t) = \frac{\lambda^2}{32\pi^2} \left( \log \frac{\Lambda_{\text{HD}}^2}{m^2} - 2 + I \left( \frac{t}{m^2} \right) \right) \]  \hspace{1cm} (S.41)

becomes identical with eqs. (S.18) and (S.17) when we identify

\[ \log \Lambda_{\text{HD}}^2 - 2 \equiv \log \Lambda_{\text{PV}}^2 \equiv \log \Lambda_{\text{HE}}^2 - 1, \]  \hspace{1cm} (S.42)

or equivalently

\[ \Lambda_{\text{hard edge}}^2 \equiv \Lambda_{\text{Pauli–Villars}}^2 \times \exp(+1) \equiv \Lambda_{\text{higher derivative}}^2 \times \exp(-1). \]  \hspace{1cm} (2)