Problem 1(a):

At the tree level, the scalar decay amplitude is simply
\[ i\mathcal{M}(S \to f + \bar{f}) = \bar{u}_f (-ig) v_{\bar{f}}. \] (S.1)

Summing over spins of the outgoing fermions, we get
\[ \sum |\mathcal{M}|^2 = g^2 \times \text{tr} [(\hat{p}_1 + m_f)(\hat{p}_2 - m_f)] = g^2 \times (4p_1p_2 - 4m^2) = 2g^2 \times (M^2_s - 4m^2), \] (S.2)

where the last equality follows from \( p_1^2 = p_2^2 = m_f^2 \) and \( (p_1 + p_2)^2 = p_s^2 = M_s^2 \).

The phase space factor for one particle decaying into two is (in the frame of the original particle)
\[ d\mathcal{P} = \frac{1}{2M_s} \times \frac{d^3\mathbf{p}}{(2\pi)^3} \times (2\pi)\delta(E_1 + E_2 - M_s) = \frac{|\mathbf{p}|}{32\pi^2 M_s^2} \times d\Omega_{\mathbf{p}} \quad \Rightarrow \quad \mathcal{P} = \frac{|\mathbf{p}|}{16\pi M_s^2}, \] (S.3)

so the tree-level decay rate is
\[ \Gamma(S \to f + \bar{f}) = \mathcal{P} \times \sum |\mathcal{M}|^2 = \frac{g^2}{8\pi} \times \frac{M^2_s - 4m_f^2}{M_s^2} \times |\mathbf{p}|. \] (S.4)

Here \( \pm \mathbf{p} \) denotes momenta of the outgoing fermions. By energy conservation,
\[ |\mathbf{p}| = \sqrt{\left(\frac{1}{2} M_s\right)^2 - m_f^2} = \frac{\beta M_s}{2} \quad \text{where} \quad \beta = \sqrt{1 - \frac{4m_f^2}{M_s^2}}. \] (S.5)

In terms of the fermions’ speed \( \beta \),
\[ \Gamma^{\text{tree}}(S \to f + \bar{f}) = \frac{g^2}{16\pi} \times \beta^3 M_s. \] (S.6)

Note that for weak Yukawa coupling \( \frac{g^2}{16\pi} \ll 1 \), the decay rate is small compared to the scalar’s mass, \( \Gamma \ll M_s \), so the resonance to to the unstable scalar should be narrow.
Problem 1(b):
In class we have calculated
\[
\Sigma_{\phi}^{1\text{loop}}(p^2) = \frac{12g^2}{16\pi^2} \int_0^1 dx \Delta(x) \times \left( \frac{1}{\epsilon} + \frac{1}{3} - \gamma_E + \log \frac{4\pi\mu^2}{\Delta(x)} \right) \tag{S.7}
\]
where \(\Delta(x) = m_f^2 - x(1-x)p^2\). For \(p^2 = \text{real } \pm i\epsilon\), everything in eq. (S.7) is real, except for the logarithm when \(\Delta < 0\):
\[
\text{Im } \log \frac{4\pi\mu^2}{\Delta(x)} = -\text{Im } \log(\Delta - i\epsilon) = +\pi \times \Theta(\Delta < 0). \tag{S.8}
\]
Consequently, the imaginary part of \(\Sigma_{\phi}\) is given by
\[
\text{Im } \Sigma_{\phi}^{1\text{loop}}(p^2 + i\epsilon) = \frac{12g^2}{16\pi} \times \int_0^1 dx (m_f^2 - x(1-x)p^2) \times \Theta(m_f^2 - x(1-x)p^2 < 0). \tag{S.9}
\]
Technically, the \(m_f\) here is the bare fermion mass, but at the \(O(g^2)\) level of accuracy we may neglect the difference between \(m_f^{\text{bare}}\) and \(m_f^{\text{phys}}\). Consequently, the threshold for the imaginary part (S.9) lies at \(p^2_{\text{min}} = (2m_f^{\text{phys}})^2\) — which is precisely the lowest scalar mass \((M_s^{\text{phys}})^2\) that allows for decay \(S \rightarrow f + \bar{f}\).

Letting \(p^2 = M_s^2 > 4m_f^2\), we have
\[
\frac{m_f^2}{p^2} = \frac{1 - \beta^2}{4} \implies \Delta(x) = \frac{M_s^2}{4} \times ((1-2x)^2 - \beta^2) \tag{S.10}
\]
— which becomes negative for \(\frac{1-\beta}{2} < x < \frac{1+\beta}{2}\). Consequently,
\[
\text{Im } \Sigma_{\phi}^{1\text{loop}}(M_s^2 + i\epsilon) = \frac{3g^2M_s^2}{16\pi} \times \int_{1/2}^{1+\beta} dx ((1-2x)^2 - \beta^2) = \frac{3g^2M_s^2}{16\pi} \times \frac{2\beta^3}{3} = -\frac{g^2}{8\pi} \times \beta^3 M_s^2. \tag{S.11}
\]
Problem 1(c):
By inspection of eqs. (S.6) and (S.11), eq. (1) holds true:
\[ \text{Im } \Sigma^{1\text{loop}}(p^2 = M_s^2 + i\epsilon) = -\frac{g^2}{8\pi} \times \beta^3 M_s^2 = -M_s \times \Gamma^{\text{tree}}(S \rightarrow f + \bar{f}). \] (1)

Higher-loop imaginary parts are similarly related to the decay rates calculated to higher orders. In the bare perturbation theory (using the bare \( \lambda_b \) and \( M_b^2 \) parameters and \( Z \) factors instead of the counterterms),
\[ \text{Im } \Sigma^\text{bare pert. theory}(p^2 = (M_s^{\text{phys}})^2 + i\epsilon) = -M_s^{\text{phys}} \times \Gamma_{\text{total}}(S \rightarrow \text{anything}) \times Z_\phi; \] (S.12)
in the perturbation theory using counterterms, the \( \Sigma_\phi(p^2) \) amplitude has a different normalization by a \( 1/Z_\phi \) factor, so we have simply
\[ \text{Im } \Sigma^\text{counterterm pert. theory}(p^2 = (M_s^{\text{phys}})^2 + i\epsilon) = -M_s^{\text{phys}} \times \Gamma_{\text{total}}(S \rightarrow \text{anything}). \] (S.13)

Eqs. (S.12) and (S.13) work in all quantum field theories. For any field \( \hat{\phi}(x) \) which can create an unstable particle \( U \) of physical mass \( M_U \) and lifetime \( 1/\Gamma_U \gg 1/M_U \), the imaginary part of \( \Sigma_\phi \) for that field satisfies
\[ \begin{align*}
\text{Im } \Sigma^\text{bare pert. theory}(p^2 = (M_U^{\text{phys}})^2 + i\epsilon) & = -M_U^{\text{phys}} \times \Gamma_{\text{total}}(U \rightarrow \text{anything}) \times Z_\phi, \\
\text{Im } \Sigma^\text{counterterm pert. theory}(p^2 = (M_U^{\text{phys}})^2 + i\epsilon) & = -M_U^{\text{phys}} \times \Gamma_{\text{total}}(U \rightarrow \text{anything}).
\end{align*} \] (S.14)

The relation (S.14) follows from the optical theorem, which makes a narrow resonance out of any slowly-decaying particle. Consequently, the propagator of the field creating such particles should have form
\[ G_{\phi\phi}(p^2 + i\epsilon) = \frac{iZ}{p^2 - (M_U^{\text{phys}})^2 + iM_U^{\text{phys}} \times \Gamma_{\text{total}}(U \rightarrow \text{anything})} + \text{finite} \] (S.15)
for \( p^2 \) near \( (M_U^{\text{phys}})^2 \). In perturbation theory, the propagator is
\[ G_{\phi\phi}(p^2) = \frac{i}{p^2 - m_{\text{bare}}^2 - \Sigma_\phi(p^2)}. \] (S.16)
To make a Breit–Wigner resonance (S.15) out of this formula, we need

\[
(M_U^{\text{phys}})^2 - (m_\phi^{\text{bare}})^2 = \text{Re} \Sigma_\phi(p^2 = (M_U^{\text{phys}})^2 + i\epsilon),
\]

(S.17)

\[
\frac{1}{Z_\phi} = 1 - \text{Re} \frac{d\Sigma_\phi}{dp^2}|_{p^2=(M_U^{\text{phys}})^2+i\epsilon},
\]

(S.18)

\[
\text{Im} \Sigma_\phi(p^2 = (M_U^{\text{phys}})^2 + i\epsilon) < 0 \quad \text{(this is essential!)}
\]

(S.19)

\[
M_U^{\text{phys}} \times \Gamma_{\text{tot}}(U \rightarrow \text{anything}) \times Z_\phi = -\text{Im} \Sigma_\phi(p^2 = (M_U^{\text{phys}})^2 + i\epsilon).
\]

(S.20)

In addition, we also assume that \(\Gamma_{\text{tot}}(U) \ll M_U^{\text{phys}}\) and that the imaginary part \(\text{Im} \Sigma_\phi(p^2 + i\epsilon)\) does not change much for \(p^2 = (M_U^{\text{phys}})^2 \pm O(M_U^{\text{phys}} \times \Gamma_{\text{tot}}(U))\). If these assumptions fail, the resonance looks wide and/or deformed rather than a nice Breit–Wigner peak (S.15).

**Problem 2. Feynman tricks:**

Let’s start by generating a whole series of Feynman parameter tricks for simplifying propagator product. In general, we have some product of denominators like \((1/A) \times (1/B)\) and we re-write it as

\[
\frac{1}{A} \times \frac{1}{B} = \int_0^1 dx \frac{1}{[(1-x)A + xB]^2}
\]

(S.21)

We have used this formula in class, and it can be easily verified by performing the integral. But there is a whole series of similar formulae which obtain by taking derivatives \((\partial/\partial A)^m(\partial/\partial B)^n\) of both sides of eq. (S.21). In this way,

\[
\frac{(-1)^{m+n}m!n!}{A^{m+1}B^{n+1}} = \int_0^1 dx \frac{(-1)^{m+n}(m+n)! \times (1-x)^m x^n}{[(1-x)A + xB]^{2+m+n}}
\]

(S.22)

and hence

\[
\frac{1}{A^{m+1}} \times \frac{1}{B^{n+1}} = \frac{(m+n+1)!}{m!n!} \times \int_0^1 dx \frac{(1-x)^m x^n}{[(1-x)A + xB]^{2+m+n}}.
\]

(S.23)

For example,

\[
\frac{1}{A} \times \frac{1}{B^2} = \int_0^1 dx \frac{2x}{[(1-x)A + xB]^3},
\]

(S.24)
\[
\frac{1}{A} \times \frac{1}{B^{n+1}} = \int_0^1 dx \frac{(n+1)x^n}{[1 - x]A + xB]^{2+n}}, \quad (S.25)
\]

\[
\frac{1}{A^2} \times \frac{1}{B^2} = \int_0^1 dx \frac{6x(1-x)}{[1 - x]A + xB]^{4}}, \quad (S.26)
\]

etc., etc.

There are also formulae for products of three or more distinct denominators. For example,

\[
\frac{1}{A} \times \frac{1}{B} \times \frac{1}{C} = \frac{1}{C} \times \int_0^1 d\xi \frac{1}{[(1 - \xi)A + \xi B]^2} \times \frac{1}{C}
\]

\[
= \int_0^1 d\xi \int_0^{2w} dw \frac{1}{[(1 - w)A + wB + \xi B]^3}
\]

\[
= \int_0^1 dz \int_0^{1-z} dx \frac{2}{[xA + (1-x-z)B + zC]^3}
\]

where on the last line I have changed integration variables from \(\xi\) and \(w\) to \(z = 1 - w\) and \(x = w \times (1 - \xi)\). This formula can be made symmetric by introducing a third variable \(y = 1 - x - z = w \times \xi\):

\[
\frac{1}{A} \times \frac{1}{B} \times \frac{1}{C} = \int \int \int dx\, dy\, dz \, \delta(x+y+z-1) \times \frac{2}{[xA+yB+zC]^3}, \quad (S.28)
\]

cf. eq. (3). By taking derivatives of both sides of this formula with respect to \(A\), \(B\), and \(C\), we get more formulae for three denominators with non-trivial powers, generally

\[
\frac{1}{A^\ell} \times \frac{1}{B^m} \times \frac{1}{C^n} = \frac{(\ell + m + n + 2)!}{\ell!\, m!\, n!} \times \int \int \int dx\, dy\, dz \, \delta(x+y+z-1) \times \frac{x^\ell\, y^m\, z^n}{[xA+yB+zC]^{\ell+m+n+3}}, \quad (S.29)
\]

5
And at this point, it’s easy to generalize to products of any number of denominators,

\[
\prod_{i=1}^{N} \frac{1}{A_i} = \int \cdots \int d^N x \delta(x_1 + \cdots + x_N - 1) \times \frac{(N-1)!}{[x_1A_1 + \cdots + x_NA_N]^N},
\]

(S.30)

and hence to any number of denominators with non-trivial powers,

\[
\prod_{i=1}^{N} \frac{1}{A_i^{\ell_i}} = \frac{(N+L-1)!}{\ell_1! \cdots \ell_N!} \times \int \cdots \int d^N x \delta(x_1 + \cdots + x_N - 1) \times \frac{1}{[x_1A_1 + \cdots + x_NA_N]^{N+L}}
\]

(S.31)

where \( L = \ell_1 + \cdots + \ell_N \).

Problem 2, the diagram:

Now that we have learned a few Feynman tricks for combining multiple denominators, we may use them for the three propagators of the two-loop diagram (2). Applying f-la (S.28) to the propagators, we immediately obtain eq. (3) and hence

\[
\Sigma^{2\text{loop}}(p^2) = -\frac{\lambda^2}{6} \int \int \int dxdydz \delta(x + y + z - 1) \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{2}{[D]^3}
\]

(S.32)

where \( D \) is given by eq. (4) in terms of \( x, y, z \) and momenta \( q_1, q_2, \) and \( q_3 \equiv p - q_1 - q_2 \).

What we need to do now is to shift the independent momentum variables from \( q_1 \) and \( q_2 \) to some \( k_1 \) and \( k_2 \) so that \( D \) takes sum-of-squares form (5). So let us start by expanding the \( z(q_3 = p - q_1 - q_2)^2 \) term in \( D \) and then let’s collect all the terms containing the \( q_1 \) momentum into a full square,

\[
D + m^2 = xq_1^2 + yq_2^2 + z(p - q_1 - q_2)^2
\]

\[
= (x + z)q_1^2 + 2zq_1^\mu(q_2 - p)_\mu + z(q_2 - p)^2 + yq_2^2
\]

\[
= (x + z) \left( q_1 + \frac{z}{x+z} (q_2 - p) \right)^2 + \frac{xz}{x+z} (q_2 - p)^2 + yq_2^2.
\]

(S.33)

Naturally, we interpret the first term on the last line as \( \alpha k_1^2 \), thus

\[
\alpha = (x + z), \quad k_1 = q_1 + \frac{z}{x+z} (q_2 - p).
\]

(S.34)

For the other two terms on the last line, we expand \( (q_2 - p)^2 \) and collect all terms containing
the $q_2$ momentum into another full square, thus

$$\frac{xz}{x+z} (q_2 - p)^2 + yq_2^2 = \frac{xz + y(x+z)}{x+z} \left( q_2 - \frac{xz}{xz + y(x+z)} p \right)^2 + \frac{xzy}{xz + y(x+z)} p^2. \quad (S.35)$$

Consequently, we define

$$\beta = \frac{xy + xz + yz}{x+z}, \quad \gamma = \frac{xyz}{xy + xz + yz}, \quad k_2 = q_2 - \frac{xz}{xy + xz + yz} p, \quad (S.36)$$

which makes the right hand side of eq. (S.35) into $\beta k_2^2 + \gamma p^2$. Altogether, we have

$$xq_1^2 + yq_2^2 + zq_3^2 = \alpha k_1^2 + \beta k_2^2 + \gamma p^2 \quad (S.37)$$

and hence eq. (5).

Finally, we need to check the Jacobian of replacing the original independent loop momenta $q_1$ and $q_2$ with $k_1$ and $k_2$. In light of eqs. (S.34) and (S.36), it is easy to see that

$$\frac{\partial (k_1, k_2)}{\partial (q_1, q_2)} = \det \begin{pmatrix} 1 & \frac{z}{x+z} \\ 0 & 1 \end{pmatrix} = 1, \quad (S.38)$$

and therefore $dk_1 dk_2 = dq_1 dq_2$, dimension by dimension. In other words, for fixed Feynman parameters,

$$\int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} = \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4}. \quad (S.39)$$

**Problem 3:**
Combining eqs. (S.32) and (5), we have

$$\Sigma^{\text{2-loop}}(p^2) = -\frac{\lambda^2}{6} \iint_{x,y,z>0} dx \, dy \, dz \, \delta(x + y + z - 1) \times$$

$$\times \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} [D = \alpha k_1^2 + \beta k_2^2 + \gamma p^2 - m^2 + i\epsilon]^3. \quad (S.40)$$
The momentum integral here has form

\[ \int \frac{d^8 k}{[k^2 + \ldots]^3} \]  

which has a quadratic divergence for \( k \to \infty \), and that creates all kinds of troubles, especially for the momentum regularization we shall use in part 3.

Now let’s take \( d/dp^2 \) derivatives of both sides of eq. \( \text{(S.40)} \). On the right hand side, the only thing which depends on \( p^2 \) is the \( \gamma p^2 \) term in \( \mathcal{D} \), everything else is completely independent on the external momentum \( p \). Hence,

\[
\frac{d\Sigma}{dp^2} = -\frac{\lambda^2}{6} \int \int \int dx \, dy \, dz \, \delta(x + y + z - 1) \times \\
\times \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \left\{ \frac{\partial}{\partial p^2} \left( \frac{2}{\mathcal{D}^4} \right) = -\frac{6\gamma}{\mathcal{D}^4} \right\},
\]

cf. eq. \( \text{(7)} \). Here, the momentum integral has form

\[ \int \frac{d^8 k}{[k^2 + \ldots]^4} \]

so the UV divergence for \( k \to \infty \) is logarithmic rather than quadratic.

**Problem 4:**
Rotating both loop momenta \( k_1 \) and \( k_2 \) into Euclidean momentum space, we have \( d^4 k_1 \to id^4 k_1^E \), \( d^4 k_2 \to id^4 k_2^E \), and

\[ \mathcal{D} \to -\alpha(k_1^E)^2 - \beta(k_2^E)^2 + \gamma p^2 - m^2, \]

hence

\[
\frac{d\Sigma}{dp^2} = +\frac{\lambda^2}{6} \int \int dx \, dy \, dz \, \delta(x + y + z - 1) \int \frac{d^4 k_1^E}{(2\pi)^4} \int \frac{d^4 k_2^E}{(2\pi)^4} \frac{6 \times (-\gamma)}{[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2]^4}.
\]

Next, we need dimensional regularization to actually perform the momentum integrals.
Changing
\[
\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \rightarrow \mu^{2(4-D)} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D}
\]
(Euclidean signature for all dimensions), we have
\[
\mu^{8-2D} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \frac{6}{[\alpha(k^E_1)^2 + \beta(k^E_2)^2 + m^2 - \gamma p^2]^4} =
\]
\[
\langle \text{using eq. (9)} \rangle
\]
\[
= \mu^{8-2D} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \int_0^\infty dt t^3 \exp \left(-t[\alpha(k^E_1)^2 + \beta(k^E_2)^2 + m^2 - \gamma p^2]\right)
\]
\[
= \mu^{8-2D} \int_0^\infty dt t^3 e^{-t(m^2-\gamma p^2)} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} e^{-t\alpha k_1^2} e^{-t\beta k_2^2}
\]
\[
\langle \text{using eq. (10)} \rangle
\]
\[
= \mu^{8-2D} \int_0^\infty dt t^3 e^{-t(m^2-\gamma p^2)} \times (4\pi \alpha t)^{-D/2} (4\pi \beta t)^{-D/2}
\]
\[
= \frac{\mu^{8-2D}}{(4\pi)^D(\alpha \beta)^{D/2}} \times \int_0^\infty dt t^{3-D} e^{-t(m^2-\gamma p^2)}
\]
\[
= \frac{\mu^{8-2D}}{(4\pi)^D(\alpha \beta)^{D/2}} \times \Gamma(4-D)(m^2 - \gamma p^2)^{D-4}.
\]
\[
\text{(S.47)}
\]
Note the $\Gamma(4-D)$ factor: It has a pole at $D = 4$ but no poles at $D < 4$. This is dimensional regularization’s way to show that the momentum integrals diverge, but only logarithmically.

At this point, we may take $D = 4 - 2\epsilon$ for an infinitesimally small $\epsilon$. Hence, the last line of eq. (S.47) becomes
\[
\frac{1}{(4\pi)^4(\alpha \beta)^2} \Gamma(2\epsilon) \left(\frac{4\pi \mu^2 \sqrt{\alpha \beta}}{m^2 - \gamma p^2}\right)^{2\epsilon} \rightarrow \frac{1}{(4\pi)^4(\alpha \beta)^2} \times \left(\frac{1}{2\epsilon} - \gamma E + \log \frac{4\pi \mu^2 \sqrt{\alpha \beta}}{m^2 - \gamma p^2}\right).
\]
\[
\text{(S.48)}
\]
Plugging this formula back into eq. (S.45) and assembling all the factors, we finally arrive at

\[
\frac{d\Sigma}{dp^2} = -\frac{\lambda^2}{3072\pi^4} \int\int dx\,dy\,dz \, \delta(x+y+z-1) \frac{\gamma}{(\alpha\beta)^2} \times \left\{ \frac{1}{\varepsilon} + 2 \log \frac{\mu^2}{m^2} + C \alpha\beta + \log \frac{\alpha\beta}{1 - (p^2/m^2)\gamma^2} \right\} \tag{S.49}
\]

where \( C = 2 \log(4\pi) - 2\gamma_E \) is a numerical constant while \( \alpha(x, y, z), \beta(x, y, z) \) and \( \gamma(x, y, z) \) depend on the Feynman parameters according to eqs. (S.34) and (S.36). Clearly, eq. (S.49) has form (8) where

\[
F(x, y, z) = -\frac{\lambda^2}{3072\pi^4} \times \frac{\gamma}{(\alpha\beta)^2} = -\frac{\lambda^2}{3072\pi^4} \times \frac{xyz}{(xy + xz + yz)^3} \tag{S.50}
\]

and

\[
G(x, y, z; p^2/m^2) = \frac{\alpha\beta}{[1 - (p^2/m^2)\gamma^2]^2} = \frac{(xy + xz + yz)^3 m^4}{[(xy + xz + yz)m^2 - xyzp^2]^2}. \tag{S.51}
\]

**Problem 5:**

When a divergent diagram is regularized using DR (dimensional regularization), the \( 1/\varepsilon \) poles could come from several places. Most commonly, they appear as \( \Gamma(\varepsilon) \) or \( \Gamma(2\varepsilon) \) factors from integrals over \( t \)-like parameters introduced to make the momentum integral Gaussian, for example see the last couple of lines of eq. (S.47). But for some diagrams — especially with nested or overlapping divergences, see §10.5 of the textbook for an example — there are additional singularities for \( \varepsilon \to 0 \) coming from divergent integrals over the Feynman parameters.

Fortunately, this does not happen for the two-loop amplitude in question, and that’s what we need to verify in this part of the problem.

We have 3 Feynman parameters \( x, y, z \) satisfying \( x, y, z \geq 0 \) and \( x + y + z = 1 \); together,
they span a 2D area (since only 2 are independent) in the shape of an equilateral triangle

![Diagram of an equilateral triangle with axes labeled](S.52)

We are to verify that functions \( F(x, y, z) \) and \( F(x, y, z) \times \log G(x, y, z) \) maybe safely integrated over that area, so let’s start with

\[
F(x, y, z) = \text{const} \times \frac{xyz}{(xy + xz + yz)^3}
\]  

and check it for singularities. The denominator \((xy + xz + yz)^3\) stays positive in the interior of the triangle (green area in fig. (S.52) where all three of \( x, y, z \) are positive) and also along the edges (blue lines where precisely one of \( x, y, z \) becomes zero), but it vanishes in the vertices (red dots where two variables go to zero at the same time). So as far as integral (12) is concerned, the only potentially dangerous parts of the triangle are the vertices, all other places are completely safe.

Let’s take a closer look at any one vertex (they are related by symmetry), say \( x, y \to 0 \) where \( z \approx 1 \). Near this vertex

\[
F \propto \frac{xy}{(x + y)^3},
\]

and if we approach this vertex along a line \( y = x \times \) a constant, then

\[
F \propto \frac{1}{x} \to \infty \text{ as } x \to 0.
\]

This behavior would create a divergence in one-dimensional integral \( \int dx \), but not for the 2D integral we are interested in. Indeed, let’s change our coordinates near the vertex from \( x, y \geq 0 \)

\[
\]
to \( w = x + y \) and \( \xi = x/w \), thus

\[
x = \xi \times w, \quad y = (1 - \xi) \times w, \quad q \geq 1 \text{ while } 0 \leq \xi \leq 1, \quad dx \, dy = w \, dw \, dx.
\] (S.56)

Consequently,

\[
F \propto \frac{\xi (1 - \xi)}{w}
\] (S.57)

but the differential

\[
F \, dx \, dy = F \times w \, dw \, d\xi \propto \xi (1 - \xi) \, d\xi \times dw,
\] (S.58)

remains perfectly finite for \( w \to 0 \) and the integral converges just fine.

Now consider the other integral (13) where we have an extra \( \log G(x, y, z) \) factor in the integrand. Since \( G \) is a rational function, \( \log G \) does not have any singularities worse that logarithmic, and log singularities may be safely integrated over. The only potential danger comes from singularities of the \( \log G \) coinciding with singularities of the \( F \) factor, so the net singularity becomes worse.

Since \( F \)'s singularities lie at the 3 corner of the triangle, let's see how the \( G \) function ant its log behave hear the corners. Going back to the \( x, y \to 0, z \approx 1 \) corner, we have

\[
G \approx \frac{(x + y)^3 m^4}{[(x + y)m^2 - xyp^2]^2} \approx (x + y)
\] (S.59)

so \( \log G \) has a logarithmic singularity on top of the “pole” of \( F \). However, in terms of the \( w, \xi \) coordinates, the differential

\[
F \times \log G \times dx \, dy \propto \xi (1 - \xi) \, d\xi \times \log(w) \, dw
\] (S.60)

has only a mild logarithmic singularity at \( w \to 0 \) and the integral converges.
Problem 6:
Having verified that the integral (S.49) over the Feynman parameters converges, we now face
the daunting task of actually evaluating the integral. Fortunately, we do not need to evaluate
its as an analytic function of the external momentum \( p^2 \) — for the purpose of calculating
the field strength renormalization factor \( Z \) we are interested in only one value of \( p^2 \), namely
\( p^2 = \text{physical mass}^2 \). Moreover, since we are working at the leading order of perturbation theory
which contributes to the \( d\Sigma/dp^2 \), we may neglect the difference between the physical and the
bare masses as higher-order correction and set \( p^2 = m^2 \). Consequently, eq. (S.49) simplifies to

\[
\left. \frac{d\Sigma}{dp^2} \right|_{p^2=m^2} = -\frac{\lambda^2}{3072\pi^4} \int\!\!\!\int\!\!\!\int dx \, dy \, dz \, \delta(x + y + z - 1) \frac{\gamma}{(\alpha\beta)^2} \times \left\{ \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C \right\} \\
\times \left\{ \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C + \log \left( \frac{(xy + xz + yz)^3}{(xy + xz + yz - xyz)^2} \right) \right\}
\]

(S.61)
where the second equality follows from eqs. (S.34) and (S.36) for the \( \alpha(x, y, z) \), \( \beta(x, y, z) \), and
\( \gamma(x, y, z) \).

Despite the above simplification, eq. (S.61) is a painful mess to evaluate. And that’s why I
gave you eqs. (18) which tell you what the integrals actually are. Using eqs. (18), we find

\[
\left. \frac{d\Sigma}{dp^2} \right|_{p^2=m^2} = -\frac{\lambda^2}{6144\pi^4} \left\{ \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C - \frac{3}{2} \right\}
\]

(S.62)
to the leading order in \( \lambda \), and therefore

\[
Z = \left. \frac{1}{1 - \frac{d\Sigma}{dp^2}} \right|_{p^2=M^2} = 1 + \frac{\lambda^2}{6144\pi^4} \left\{ \frac{1}{\epsilon} + 2 \log \frac{\mu^2}{m^2} + C - \frac{3}{2} \right\} + O(\lambda^3).
\]

(S.63)