PHY–396 L. Solutions for homework set #20.

Problem 1(a):
As explained in class, at high momenta \( p^2 \gg m^2 \) we may approximate the electron’s propagator as

\[
\frac{i}{\not{p} - m + i0} = \frac{i(p + m)}{p^2 - m^2 + i0} \approx \frac{i(p + m)}{p^2 + i0}
\]  
(S.1)

The \( m^2 \) term in the denominator becomes negligible at high energies, but the \( m \) term in the numerator remains important (for some processes) because it changes the electron helicity (in the context of propagator \( \times \) vertex). In other words, at high energies \( m \) acts as a valence = 2 coupling between the left and right chiralities of the electron, but its role as a mass is not important. Consequently, \( m(E) \) renormalizes like all the other couplings a QFT.

Specifically, the renormalized mass \( m(E) \) is related to the bare mass \( m_b \) according to

\[
m_b = \frac{m(E) + \delta_m(E)}{1 + \delta_2(E)}.
\]  
(S.2)

Since the bare mass does not depend on the renormalization point, the renormalized mass and the counterterms satisfy

\[
\frac{dm}{d\log E} = (m + \delta_m) \times \frac{d\delta_2}{d\log E} - \frac{d\delta m}{d\log E} = (m + \delta_m) \times 2\gamma_e - \frac{d\delta m}{d\log E}.
\]  
(S.3)

At the one-loop level this formula simplifies to

\[
\frac{dm}{d\log E} = 2m\gamma_2 - \frac{d\delta m}{d\log E}.
\]  
(S.4)

In QED, the \( \delta_m \) counterterm is proportional to the electron’s mass itself,

\[
\delta m(E) = m \times \hat{\delta}(E),
\]  
(S.5)

because for \( m = 0 \) the theory has a chiral symmetry which leads to \( \delta m = 0 \). Plugging eq. (S.5)
into eq. (S.3) we get

\[
\frac{dm}{d\log E} = 2m(1 + \hat{\delta})\gamma_e - m \frac{d\hat{\delta}}{d\log E},
\]  

(S.6)

or equivalently

\[
\frac{dm}{d\log E} = m \times \gamma_m(\alpha(E))
\]  

(1)

where

\[
\gamma_m = 2\gamma_e \times (1 + \hat{\delta}) - \frac{d\hat{\delta}}{d\log E}.
\]  

(S.7)

In the Minimal Subtraction regularization scheme the counterterms generally look like

\[
\delta_1(\epsilon, \alpha) = \delta_2(\epsilon, \alpha) = \frac{C_2(\alpha)}{\epsilon} + \text{higher poles},
\]

\[
\delta_3(\epsilon, \alpha) = \frac{C_3(\alpha)}{\epsilon} + \text{higher poles},
\]  

(S.8)

\[
\hat{\delta}(\epsilon, \alpha) = \frac{\hat{C}(\alpha)}{\epsilon} + \text{higher poles},
\]

In terms of such counterterms, the anomalous dimension of the electron field \(\Psi\) is

\[
\gamma_e(\alpha) = -\alpha \frac{d}{d\alpha} C_2(\alpha)
\]  

(S.9)

while the anomalous dimension (S.7) of the electron’s mass becomes

\[
\gamma_m(\alpha) = \alpha \frac{d}{\alpha} (\hat{C} - C_2).
\]  

(S.10)
Problem 1(b):
The $\delta_2$ and $\delta_m$ counterterms of QED cancel the divergences of the electron self-energy correction $\Sigma(p)$. At the one-loop level, the self-energy correction comes from a single diagram

![Diagram]

which yields

$$-i \Sigma^{1\text{loop}} (p) = \int \frac{d^4k}{(2\pi)^4} \frac{i e \gamma_\mu}{k + p - m + i0} \times \frac{i e \gamma_\nu}{k^2 + i0} \left( g^{\mu\nu} + (\xi - 1) \frac{k^\mu k^\nu}{k^2} \right). \quad (S.12)$$

Note that we do not fix the Feynman gauge here but allow for a general gauge parameter $\xi$ for the photon propagator $(2)$.

For large loop momentum $k \gg p, m$ we may expand the fermion propagator in powers of $(m - p)/k$,

$$\frac{1}{k + p - m + i0} = \frac{1}{k + i0} + \frac{1}{k + i0} (m - p) \frac{1}{k + i0} + \frac{1}{k + i0} (m - p) \frac{1}{k + i0} (m - p) \frac{1}{k + i0} + \cdots. \quad (S.13)$$

Only the first two terms in this expansion contribute to the UV divergence of the integral $(S.12)$, thus

$$\Sigma^{1\text{loop}}_{\text{div}} (p) = -i e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i0} \left( g^{\mu\nu} + (\xi - 1) \frac{k^\mu k^\nu}{k^2} \right) \times$$

$$\times \gamma_\mu \left( \frac{1}{k + i0} + \frac{1}{k + i0} (m - p) \frac{1}{k + i0} \right) \gamma_\nu, \quad (S.14)$$

$$\Sigma^{1\text{loop}} (p) = \Sigma^{1\text{loop}}_{\text{div}} (p) + \text{finite}(p).$$

On the second line here, we have

$$\gamma_\mu \left( \frac{1}{k + i0} + \frac{1}{k + i0} (m - p) \frac{1}{k + i0} \right) \gamma_\nu = \frac{\gamma_\mu k^\nu}{k^2 + i0} + \frac{\gamma_\mu k (m - p) k^\nu}{[k^2 + i0]^2}. \quad (S.15)$$
Multiplying this expression by the photon propagator (2), we obtain

\[
\text{integrand} = \frac{\gamma_\mu \gamma_\mu}{k^2 + i0} + \frac{\gamma_\mu (m - \not{p}) \gamma_\mu}{k^2 + i0} + (\xi - 1) \frac{k \cdot \not{k}}{k^2 + i0} + (\xi - 1) \frac{k \cdot (m - \not{p}) \cdot \not{k}}{k^2 + i0} \\
= -2 \frac{k}{k^2 + i0} + 4m \frac{k^2}{k^2 + i0} + (\xi - 1) \frac{k}{k^2 + i0} + (\xi - 1) \frac{m - \not{p}}{k^2 + i0} \\
= (\xi - 3) \frac{k}{k^2 + i0} + (\xi + 3) \frac{m}{k^2 + i0} + (1 - \xi) \frac{\not{p}}{k^2 + i0} + 2 \not{k} \not{p} \not{k} \\
\text{(S.16)}
\]

Moreover, in the context of a Lorentz-invariant momentum integral, the first term on the bottom line here integrates to zero, while in the numerator of the last term \(k^\mu k^\nu \approx g^\mu\nu k^2/4\) and hence

\[
2 \not{k} \not{p} \not{k} = 4(kp) \not{k} - 2k^2 \not{p} \approx 4 \not{p} \times \frac{k^2}{4} - 2 \not{p} k^2 = -k^2 \times \not{p}. \quad \text{(S.17)}
\]

Thus,

\[
\text{integrand} \approx (\xi + 3) \frac{m}{k^2 + i0} + (1 - \xi - 1) \frac{\not{p}}{k^2 + i0} \quad \text{(S.18)}
\]

and therefore

\[
\Sigma_{\text{div}}^{1\text{loop}} = e^2 [(\xi + 3)m - \xi \not{p}] \times \int_{\text{reg}} \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 + i0} . \quad \text{(S.19)}
\]

The integral here seems to have both UV and IR divergences in 4 dimensions, but the IR divergence is an artefact of the \(1/\not{k}\) expansion (S.13) which does not work for small momenta. On the other hand, the UV divergence is genuine,

\[
\int_{\text{reg}} \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 + i0} = \frac{+1}{16\pi^2} \times \left( \frac{1}{\epsilon} + \text{const OR log} \Lambda^2 + \text{const} \right) , \quad \text{(S.20)}
\]

therefore

\[
\Sigma^{1\text{loop}}(\not{p}) = \frac{e^2}{16\pi^2} \times [(3 + \xi)m - \xi \not{p}] \times \left( \frac{1}{\epsilon} \text{OR log} \Lambda^2 \right) + \text{finite}(\not{p}) . \quad \text{(S.21)}
\]

This divergence must be canceled by the QED counterterms \(\delta_2\) and \(\delta_m\) according to

\[
\Sigma^{\text{net}}(\not{p}) = \Sigma^{\text{loops}}(\not{p}) + \delta_m - \delta_2 \times \not{p} , \quad \text{(S.22)}
\]
hence at the one-loop level

\begin{align}
\delta_m &= -\frac{\alpha}{4\pi} \times (3 + \xi)m \times \left( \frac{1}{\epsilon} \text{ or } \log \Lambda^2 \right) + \text{finite}, \quad (S.23) \\
\delta_2 &= -\frac{\alpha}{4\pi} \times \xi \times \left( \frac{1}{\epsilon} \text{ or } \log \Lambda^2 \right) + \text{finite.} \quad (S.24)
\end{align}

**Problem 1(c):**

In the MS renormalization scheme the counterterms (S.23) and (S.24) have no finite parts,

\begin{align}
\delta_2 &= \frac{1}{\epsilon} \times -\frac{\alpha \xi}{4\pi} + O(\alpha^2) \quad (S.25) \\
\delta_m &= \frac{1}{\epsilon} \times -\frac{\alpha(3 + \xi)m}{4\pi} + O(\alpha^2 m), \quad (S.26)
\end{align}

\text{i.e.,}

\begin{align}
\hat{\delta} &= \frac{1}{\epsilon} \times -\frac{\alpha(3 + \xi)}{4\pi} + O(\alpha^2). \quad (S.27)
\end{align}

Plugging these counterterms into eq. (S.10) we immediately obtain

\begin{align}
\gamma_m &= \frac{\alpha}{4\pi} \times \left[ -(3 + \xi) + \xi \right] + O(\alpha^2) = -\frac{3\alpha}{4\pi} + O(\alpha^2). \quad (S.28)
\end{align}

Note that the gauge dependence of the \( \delta_2 \) and \( \delta_m \) counterterms cancels out and the anomalous dimension (S.28) of the electron’s mass comes out to be gauge invariant.

**Problem 1(d):**

Evolution of the renormalized electron’s mass with energy is given by eq. (1). Integrating this
equation, we obtain

\[
\log \frac{m_e(M_w)}{m_e(m_e)} = \frac{\log M_w}{\log m_e} \int_{\log m_e} \gamma_m(\alpha(E)) d\log E.
\]  \quad (S.29)

At the one-loop level, the anomalous dimension of the mass is given by eq. (S.28), hence

\[
\log \frac{m_e(M_w)}{m_e(m_e)} \approx -\frac{3}{4\pi} \int_{\log m_e} \alpha(E) d\log E.
\]  \quad (S.30)

The rest of this exercise is numerics. Between the electron mass scale \( m_e = 511 \text{ keV} \) and the weak scale \( M_w \) — which we identify with the \( Z^0 \) mass \( M_Z = 91 \text{ GeV} \) — the EM coupling changes from

\[
\alpha(m_e) \approx \alpha(0) \approx \frac{1}{137.03}
\]  \quad (S.31)

to

\[
\alpha(M_z) \approx \frac{1}{129.65}
\]  \quad (S.32)

This change is only 5%, so to the first approximation we may ignore it. In other words, we approximate \( \alpha \approx \text{const} = 1/135 \) (average value), which leads to

\[
\log \frac{m_e(M_w)}{m_e(m_e)} \approx -\frac{3\alpha}{4\pi} \times \log \frac{M_Z}{m_e} \approx -0.067.
\]

Consequently,

\[
m_e(M_z) \approx m_e^{\text{phys}} \times (1 - 0.067) = 477 \text{ keV}.
\]  \quad (S.33)
Problem 3(a):
The difference between a circle and a straight line is that on a circle the path of a particle
going from point \( x_0 \) to point \( x' \) does not need to be ‘straight’ but may wrap around the whole
circle one or more times. Indeed, let us compare a particle moving on a circle according to \( x(t) \)
(modulo \( 2\pi R \)) with a particle moving on an infinite line according to \( y(t) \). If the two particles
have exactly the same velocities at all times,

\[
\frac{dx}{dt} \equiv \frac{dy}{dt} \tag{S.34}
\]

and similar initial positions \( x_0 = y_0 \) (according to some coordinate systems) at time \( t = 0 \), then
after time \( T \) one generally has

\[
y(T) = x(T) + 2\pi R \times n \tag{S.35}
\]

for some integer \( n = 0, \pm 1, \pm 2, \pm 3, \ldots \) because the \( x(y) \) path may wrap around the circle \( n \)
times while the \( y(t) \) path may not wrap. For example, the two paths depicted below have same
(constant) velocities and begin at \( y_0 = x_0 \) but end at \( y(T) = x(T) + 2\pi R \times 2 \):

It is easy to see that the paths \( x(t) \) (modulo \( 2\pi R \)) and \( y(t) \) (modulo nothing) are in one-to-one
correspondence with each other, provided we restrict the initial point \( y_0 \) of the particle on
the infinite line to a particular interval of length $L = 2\pi R$, say $0 \leq y_0 < 2\pi R$. Consequently, in the path integral for the particle on the circle

$$x(t=T) = x' \mod L \int_{x(t=0)=x_0 \mod L} D'[x(t) \mod L] = \sum_{n=-\infty}^{+\infty} \int_{y(t=0)=x_0}^{y(t=T)=x'+nL} D'[y(t)].$$ (S.36)

Furthermore, in the absence of potential energy, the circle path $x(t) \mod L$ and the corresponding infinite line path $y(t)$ have equal actions

$$S[x(t) \mod L] = S[y(t)] = \int_0^T dt \left[ \frac{M}{2} \dot{x}^2 = \frac{M}{2} \dot{y}^2 \right],$$ (S.37)

and therefore

$$U_{\text{circle}}(x'; x_0) = \int_{x(t=0)=x_0 \mod L}^{x(t=T)=x' \mod L} D'[x(t) \mod L] e^{i S[x(t) \mod L]/\hbar}$$

$$= \sum_{n=-\infty}^{+\infty} \int_{y(t=0)=x_0}^{y(t=T)=x'+nL} D'[y(t)] e^{i S[y(t)]/\hbar}$$ (1)

$$= \sum_{n=-\infty}^{+\infty} U_{\text{line}}(y' = x' + nL; y_0 = x_0).$$

Q.E.D.

Problem 3(b):
For a free particle living on an infinite line the evolution kernel is given by

$$U_{\text{line}}(y'; y_0) = \sqrt{\frac{M}{2\pi i \hbar T}} \times \exp \left( \frac{i}{\hbar} S_{\text{classical}} = \frac{i}{\hbar} \frac{M(x' - x_0)^2}{2T} \right),$$ (3)

hence according to eq. (1), a particle on a circle has kernel

$$U_{\text{circle}}(x'; x_0) = \sqrt{\frac{M}{2\pi i \hbar T}} \times \sum_{n=-\infty}^{+\infty} \exp \left( \frac{iM}{2\hbar T} (x' - x_0 + nL)^2 \right).$$ (S.38)
To evaluate this sum, we use Poisson re-summation formula (2), which gives

\[ \sum_{n=-\infty}^{+\infty} \exp \left( \frac{iM}{2\hbar T} (x' - x_0 + nL)^2 \right) = \sum_{\ell=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv \exp \left( \frac{iM}{2\hbar T} (x' - x_0 + \nu L)^2 \right) \times e^{2\pi i\nu}. \]  

(S.39)

Rearranging the exponential, we have

\[ \frac{iM}{2\hbar T} (x' - x_0 + \nu L)^2 + 2\pi i\nu = \frac{iML^2}{2\hbar T} \left( \nu + \frac{x' - x_0}{L} + \frac{2\pi \ell h T}{ML^2} \right) - 2\pi i\ell \frac{x' - x_0}{L} - \frac{i\hbar (2\pi \ell)^2}{ML^2}, \]

and therefore

\[ \int_{-\infty}^{+\infty} dv \exp \left( \frac{iM}{2\hbar T} (x' - x_0 + \nu L)^2 \right) \times e^{2\pi i\nu} = \sqrt{\frac{2\pi i\hbar T}{ML^2}} \times \exp \left( -2\pi i\ell \frac{x' - x_0}{L} - \frac{(2\pi \ell)^2 i\hbar T}{ML^2} \right). \]

(S.40)

 Consequently,

\[ U_{\text{circle}}(x'; x_0) = \sqrt{\frac{M}{2\pi i\hbar T}} \times \sqrt{\frac{2\pi i\hbar T}{ML^2}} \times \sum_{\ell=-\infty}^{+\infty} \exp \left( -2\pi i\ell \frac{x' - x_0}{L} - \frac{(2\pi \ell)^2 i\hbar T}{ML^2} \right) \]

\[ = \frac{1}{L} \sum_{\ell=-\infty}^{+\infty} e^{ip(x' - x_0)/\hbar} \times e^{-iTE/\hbar} \]

where

\[ p = -\frac{2\pi \hbar \ell}{L} = -\frac{\hbar \ell}{R} \quad \text{and} \quad E = \frac{p^2}{2M}. \]  

(S.42)

Problem 3(c): This is obvious from eqs. (S.42) and (S.43).