Problem 1(a):
As discussed in class, the quarks and the antiquarks of the B theory have the usual SQCD flavor symmetry $G_f = SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$, albeit with unusual ‘charge’ assignments: the quarks $q^c_f$ belong to the $\overline{N_f}$ multiplet of the $SU(N_f)_L$ instead of the $N_f$, and likewise the antiquarks $\bar{q}^c_f$ belong to the $N_f$ rather than the $\overline{N_f}$ of the $SU(N_f)_R$. The quantum numbers of the gauge-singlets $M^{f'f'}$ are completely determined by the invariance of the Yukawa coupling (2). Specifically, $M \in (N_f, \overline{N_f})$ of the $SU(N_f)_L \times SU(N_f)_R$, the baryon number of $M$ is zero, and the R–charge is

$$ R(M) = 2 - R(q) - R(\bar{q}) \quad \text{(S.1)} $$

because the R–charge of the superpotential is 2. Also, $M$ cannot transform under any symmetry that does not act on the other fields, which rules out any additional global symmetries of the B theory.

And here is the table of the color and flavor quantum numbers of all the elementary fields of the B theory:

<table>
<thead>
<tr>
<th>(\bar{a}, \bar{\psi}^{\alpha}_a)</th>
<th>(\psi^{\alpha}_a)</th>
<th>(a^{\alpha}, \lambda^a)</th>
<th>(a^{\mu}, \lambda^a)</th>
<th>(M, \psi^{\alpha}_M)</th>
<th>(\bar{q}, \bar{\psi}^{\alpha}_q)</th>
<th>(q, \psi^{\alpha}_q)</th>
<th>(M, \psi^{\alpha}_M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q)</td>
<td>(Q)</td>
<td>(Q)</td>
<td>(Q)</td>
<td>(Q)</td>
<td>(Q)</td>
<td>(Q)</td>
<td>(Q)</td>
</tr>
<tr>
<td>(SU(N_f)_B)</td>
<td>(SU(N_f)_L)</td>
<td>(SU(N_f)_R)</td>
<td>(B)</td>
<td>(R(\text{boson}))</td>
<td>(R(\text{fermion}))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(A^{\mu}, \lambda^a)</td>
<td>(1)</td>
<td>(1)</td>
<td>(0)</td>
<td>(0)</td>
<td>(+1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(q, \psi^{\alpha}_q)</td>
<td>(N_f^B)</td>
<td>(\overline{N}_f)</td>
<td>(+1) $N_f^B$ $1 - \frac{N_f^B}{N_f}$ $- \frac{N_f^B}{N_f}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\bar{q}, \bar{\psi}^{\alpha}_q)</td>
<td>(1)</td>
<td>(N_f)</td>
<td>(-1) $N_f^B$ $1 - \frac{N_f^B}{N_f}$ $- \frac{N_f^B}{N_f}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(M, \psi^{\alpha}_M)</td>
<td>(1)</td>
<td>(N_f)</td>
<td>(0)</td>
<td>(+2\frac{N_f^B}{N_f}) $2\frac{N_f^B}{N_f} - 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(S.2)
Similarly, for the A theory

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{QN} & SU(N^A_c) & SU(N_f)_L & SU(N_f)_R & B & R(\text{boson}) & R(\text{fermion}) \\
\hline
A^\mu, \lambda^\alpha & \text{Adjoint} & 1 & 1 & 0 & 0 & +1 \\
\hline
Q, \Psi_Q & N^A_c & N_f & 1 & + \frac{1}{N_f^2} & 1 - \frac{N^A_c}{N_f} & - \frac{N^A_c}{N_f} \\
\hline
\tilde{Q}, \tilde{\Psi}_Q & \overline{N^A_c} & 1 & N_f & - \frac{1}{N_f^2} & 1 - \frac{N^A_c}{N_f} & - \frac{N^A_c}{N_f} \\
\hline
\end{array}
\] (S.3)

**Problem 1(b):**

* First, a little group theory reminder. A triangle anomaly involving 3 generators \(T^a, T^b, T^c\) of the same non-abelian symmetry factor has form \(\text{tr}(T^a \{T^b, T^c\}) = A \times d^{abc}\) where \(d^{abc}\) is a cubic invariant of the group in question while

\[
A = \#\text{fundamentals} - \#\text{antifundamentals} + \cdots
\] (S.4)

where the \(\cdots\) stand for terms counting various tensor multiplets (which we fortunately do not have in this homework). To compare anomalies of different sets of fermions, all we need is to compare the respective coefficients \(A\).

Likewise, an anomaly involving two non-abelian generators \(T^a, T^b\) and one abelian charge \(X\) has form \(\text{tr}(T^a T^b X) = A \times \delta^{ab}\) where

\[
A = \sum_{\psi} X(\psi) \times \text{Index}(\psi) \times \#(\psi)
\] (S.5)

where the Index is with respect to the non-abelian factor in question while \# counts multiplicity with respect to the other symmetries, if any.

Finally, for a purely abelian anomaly

\[
\text{tr}(XYZ) = \sum_{\psi} X(\psi) \times Y(\psi) \times Z(\psi) \times \#\psi.
\] (S.6)
And now let’s compute and compare all the non-trivial favor anomalies of the two theories.

- The $[SU(N_f)_L]^3$ anomaly.
  The A theory has $N_c^A$ fundamentals (the quarks $\Psi_Q$) of the $SU(N_f)_L$ and no antifundamentals, thus $A(A) = N_c^A$. The B theory has $N_f$ fundamentals (the gauge singlets $\psi_M$) and $N_c^B$ antifundamentals (the quarks $\psi_q$), hence $A(B) = N_f - N_c^B = N_c^a = A(A)$.

- The $[SU(N_f)_R]^3$ anomaly: the anomalies match in the same way.

- The $[SU(N_f)_L]^2 \times U(1)_B$ anomaly.
  In the either theory, the only fermions charged under both $SU(N_f)_L$ and $U(1)_B$ symmetries are the quarks, thus

  \[
  A(A) = A(\Psi_Q) = \frac{1}{N_c^A} \times \frac{1}{2} \times N_c^A = +\frac{1}{2},
  \]

  \[
  A(B) = A(\psi_q) = \frac{1}{N_c^B} \times \frac{1}{2} \times N_c^B = +\frac{1}{2},
  \]

  and the anomalies match.

- The $[SU(N_f)_R]^2 \times U(1)_B$ anomaly: similar matching.

- The $[SU(N_f)_L]^2 \times U(1)_R$ anomaly.
  This time, in the A theory only the quarks carry both $SU(N_f)_L$ and $U(1)_R$ charges, but in the B theory, both the quarks $\psi_q$ and the gauge-singlets $\psi_M$ have the requisite quantum numbers, hence

  \[
  A(A) = A(\Psi_Q) = -\frac{N_c^A}{N_f} \times \frac{1}{2} \times N_c^A
  = -\frac{(N_c^A)^2}{2N_f},
  \]

  \[
  A(B) = A(\psi_q) + A(\psi_M)
  = -\frac{N_c^B}{N_f} \times \frac{1}{2} \times N_c^B + \left(2 \frac{N_c^B}{N_f} - 1\right) \times \frac{1}{2} \times N_f
  = -\frac{(N_c^B)^2}{2N_f} + \frac{2N_c^B - N_f}{2} = -\frac{(N_f - N_c^B)^2}{2N_f}
  \]

  and the anomalies match.
• The $[SU(N_f) R]^2 \times U(1)_R$ anomaly: similar matching.

• The abelian $[U(1)_B]^2 \times U(1)_R$ anomaly.

In both theories, this anomaly comes from the quarks and the antiquarks because the gauginos and the $\psi_M$ fermions (in B theory) have zero baryon numbers. Thus,

$$\mathcal{A}(A) = \mathcal{A}(\Psi_Q) + \mathcal{A}(\bar{\Psi}_Q)$$

$$= \left( \frac{+1}{N^A_c} \right)^2 \times \left( -\frac{N^A_c}{N_f} \right) \times N^A_c N_f + \left( \frac{-1}{N^A_c} \right)^2 \times \left( -\frac{N^A_c}{N_f} \right) \times N^A_c N_f$$

$$= -2,$$

$$\mathcal{A}(B) = \mathcal{A}(\psi_q) + \mathcal{A}(\bar{\psi}_q)$$

$$= \left( \frac{+1}{N^B_c} \right)^2 \times \left( -\frac{N^B_c}{N_f} \right) \times N^B_c N_f + \left( \frac{-1}{N^B_c} \right)^2 \times \left( -\frac{N^B_c}{N_f} \right) \times N^B_c N_f$$

$$= -2,$$

and the anomalies match.

• The abelian $[U(1)_R]^3$ anomaly.

This time we must account for all the elementary fermions of each theory since all of them have non-zero R–charges. Thus, for the A theory,

$$\mathcal{A}(A) = \mathcal{A}(\Psi_Q) + \mathcal{A}(\bar{\Psi}_Q) + \mathcal{A}(\lambda)$$

$$= \left( -\frac{N^A_c}{N_f} \right)^3 \times N^A_c N_f \times 2 + (+1)^3 \times \left( (N^A_c)^2 - 1 \right)$$

$$= -\frac{(N^A_c)^4}{N^2_f} + (N^A_c)^2 - 1,$$

while for the B theory,

$$\mathcal{A}(B) = \mathcal{A}(\psi_q) + \mathcal{A}(\bar{\psi}_q) + \mathcal{A}(\lambda) + \mathcal{A}(\psi_M)$$

$$= \left( -\frac{N^B_c}{N_f} \right)^3 \times N^B_c N_f \times 2 + (+1)^3 \times \left( (N^B_c)^2 - 1 \right) + \left( 2 \frac{N^B_c}{N_f} - 1 \right)^3 \times N^2_f$$

$$= -\frac{(N^B_c)^4}{N^2_f} + 8 \frac{(N^B_c)^3}{N_f} - 11(N^B_c)^2 + 6 N^B_c N_f - 1$$

$$= \frac{(N_f - N^B_c)^4}{N^2_f} + (N_f - N^B_c)^2 - 1,$$
and the anomalies match because \( N_f - N_c^B = N_c^A \).

- **The trace anomaly \( \text{tr}(R) \).**

Again, in each theory all fermions contribute to this anomaly, thus

\[
\mathcal{A}(A) = \mathcal{A}(\Psi_Q) + \mathcal{A}(\tilde{\Psi}_Q) + \mathcal{A}(\lambda) \\
= \left( -\frac{N_c^A}{N_f} \right) \times N_c^AN_f \times 2 + (+1) \times \left( (N_c^A)^2 - 1 \right) \\
= -(N_c^A)^2 - 1,
\]

\[
\mathcal{A}(B) = \mathcal{A}(\psi_q) + \mathcal{A}(\tilde{\psi}_q) + \mathcal{A}(\lambda) + \mathcal{A}(\psi_M) \\
= \left( -\frac{N_c^B}{N_f} \right) \times N_c^BN_f \times 2 + (+1) \times \left( (N_c^B)^2 - 1 \right) + \left( 2\frac{N_c^B}{N_f} - 1 \right) \times N_f^2 \\
= -N_f^2 + 2N_fN_c^B - (N_c^B)^2 - 1 \\
= -(N_f - N_c^B = N_c^A)^2 - 1,
\]

and the trace anomalies match too. *Quod erat demonstrandum.*

**Problem 2:**

Let’s start with the \( |J_{el}^\nu\rangle \) corresponding to the electric current \( J_{el}^\nu(x) = \partial^\mu F^{\mu\nu}(x) \). (Note: all ‘Lorentz’ indices are upper because we are in the Euclidean space.) Given \( |J_{el}^\nu\rangle = P^\mu |F^{\mu\nu}\rangle \) and hence \( \langle J_{el}^\nu |J_{el}^\nu \rangle = \langle F^{\lambda\nu} | (P^\lambda = -iK^\lambda) \rangle \), we have

\[
\langle J_{el}^\nu |J_{el}^\nu \rangle = -i \langle F^{\lambda\nu} | K^\lambda P^\mu |F^{\mu\nu}\rangle = -i \langle F^{\lambda\nu} | [K^\lambda, P^\mu] |F^{\mu\nu}\rangle
\]

since \( K^\lambda |F^{\mu\nu}\rangle = 0 \) because \( F^{\mu\nu}(x) \) is a primary operator of the conformal field theory and the corresponding state \( |F^{\mu\nu}\rangle \) is primary. Using the commutation relation

\[
-i[K^\lambda, P^\mu] = 2g^{\lambda\mu}D + 2J^\lambda \mu
\]

we obtain

\[
\langle J_{el}^\nu |J_{el}^\nu \rangle = 2 \langle F^{\lambda\nu} | (g^{\lambda\mu}D + J^{\lambda\mu} |F^{\mu\nu}\rangle
\]
where \( D\,|F^{\mu\nu}\rangle = \Delta(F)|F^{\mu\nu}\rangle \) while

\[
J^{\lambda\kappa}\,|F^{\mu\nu}\rangle = g^{\lambda\mu}\,|F^{\kappa\nu}\rangle - g^{\lambda\nu}\,|F^{\kappa\mu}\rangle - g^{\kappa\mu}|F^{\lambda\nu}\rangle + g^{\kappa\nu}|F^{\lambda\mu}\rangle \tag{S.16}
\]

and hence

\[
J^{\lambda\mu}\,|F^{\mu\nu}\rangle = g^{\lambda\mu}\,|F^{\mu\nu}\rangle - g^{\lambda\nu}\,|F^{\mu\mu}\rangle - g^{\mu\nu}|F^{\lambda\mu}\rangle = -2\,|F^{\lambda\nu}\rangle. \tag{S.17}
\]

Altogether

\[
\langle g^{\lambda\mu}D + J^{\lambda\mu}\,|F^{\mu\nu}\rangle = (\Delta - 2)|F^{\lambda\nu}\rangle \tag{S.18}
\]

and hence

\[
\langle J^{\nu}_{\text{el}}|J^{\nu}_{\text{el}}\rangle = 2(\Delta - 2)\times \langle F^{\lambda\nu}|F^{\lambda\nu}\rangle = (\Delta - 2)\times \text{positive.} \tag{S.19}
\]

Therefore, the gauge tension field \( F^{\lambda\nu} \) must have scaling dimension \( \Delta \geq 2 \), and moreover, \( \Delta = 2 \) if and only if \( |J^{\nu}_{\text{el}}\rangle \) is a null state, i.e. the electric current \( J^{\nu}_{\text{el}}(x) \) is zero in the CFT because all the electric charges are massive and decouple from the deep-IR theory.

Now consider the magnetic current \( J^{\nu}_{\text{mag}} = \partial^{\mu}\bar{F}^{\mu\nu}(x) \) and the corresponding descendant state \( |J^{\nu}_{\text{mag}}\rangle = P^{\mu}\,|\bar{F}^{\mu\nu}\rangle \). Proceeding as above, we obtain

\[
\langle J^{\nu}_{\text{mag}}|J^{\nu}_{\text{mag}}\rangle = 2\,\langle \bar{F}^{\lambda\nu}\rangle\,\langle g^{\lambda\mu}D + J^{\lambda\mu}\,|\bar{F}^{\mu\nu}\rangle \tag{S.20}
\]

and since \( \bar{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F^{\kappa\lambda} \) has the same scaling dimension \( \Delta \) and the same \( SO(4) \) ‘Lorentz’ structure — the 2-index antisymmetric tensor — as the \( F^{\mu\nu} \), we have

\[
\langle g^{\lambda\mu}D + J^{\lambda\mu}\,|\bar{F}^{\mu\nu}\rangle = (\Delta - 2)|\bar{F}^{\lambda\nu}\rangle \tag{S.21}
\]

and hence

\[
\langle J^{\nu}_{\text{mag}}|J^{\nu}_{\text{mag}}\rangle = 2(\Delta - 2)\times \langle \bar{F}^{\lambda\nu}|\bar{F}^{\lambda\nu}\rangle = (\Delta - 2)\times \text{positive.} \tag{S.22}
\]

Therefore, we have \( \Delta = 2 \) if and only if \( |J^{\nu}_{\text{mag}}\rangle \) is a null state, i.e. the magnetic current \( J^{\mu}_{\text{mag}} \) is zero in the CFT because the magnetic charges either do not exist at all or are massive and decouple from the IR theory.
Altogether, we find that either $F^{\mu\nu}(x)$ has $\Delta > 2$ and there are both electric and magnetic charges (both massless!) or else the $F^{\mu\nu}$ becomes free in the deep infrared and has $\Delta = 2$.

*Quod erat demonstrandum.*