Consider a QFT of a charged Dirac field $\Psi_c(x)$, a neutral Dirac field $\Psi_n(x)$, a charged scalar field $\Phi(x)$ and the electromagnetic field $A^\mu(x)$. Classically — and at the tree level of the quantum theory,

$$\begin{aligned}\mathcal{L} \;=\; & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; D_\mu\Phi^* D^\mu\Phi \;-\; m_s^2\Phi^*\Phi \;+\; \overline{\Psi}_c(i\not{D} - m_c)\Psi_c \;+\; \overline{\Psi}_n(i\not{\partial} - m_n)\Psi_n \\ & +\; g\Phi\overline{\Psi}_c\Psi_n \;+\; g\Phi^*\overline{\Psi}_n\Psi_c \;-\; \tfrac{1}{4}\lambda\big(\Phi^*\Phi\big)^2 .\end{aligned} \tag{1}$$

1. First, a few simple questions:
   (a) Are there any *renormalizable* couplings one may add to the Lagrangian (1) without breaking any symmetries of the theory? Explain your answer.
   (b) Write down the renormalized Lagrangian of the quantum field theory — including all the counterterms needed to cancel the divergences — and spell out the Feynman rules of the renormalized perturbation theory.
   (c) Some counterterms are related by symmetries and/or Ward–Takahashi identities. Write down all such relations.

2. Second, a lot of hard work: Calculate the infinite parts of all the independent counterterms at the one-loop order of the perturbation theory. To save time, use relations you wrote down in part 1(c): If two or more counterterms are related by a symmetry or WT identity, calculate just one of those counterterms, whichever you think is simpler.

   For each independent counterterm, start by drawing all the relevant Feynman diagrams. If a diagram was evaluated in any of the homeworks, don't waste your time copying the posted solution, just quote the result and move on to the next diagram. Ditto for the diagrams evaluated in class or in supplementary notes. For the remaining diagrams — and there will be plenty of those — use dimensional regularization and work hard.

   Many diagrams — especially those contributing to the δ_λ counterterm — are related by permutations of the external legs. Such symmetries can save you a lot of work, but please be careful counting similar diagrams.

Note: The net physical amplitudes are gauge invariant, but the individual loop diagrams and the counterterms depend on the gauge you work in. To be consistent, one must use the same gauge in all calculations. Even the diagrams you use to calculate different counterterms must be evaluated in the same gauge.

Hint: many divergences cancel out in the Landau gauge $\xi = 0$ for the photon propagators, so this is probably the best gauge to use for this problem. But if you would rather use the Feynman gauge $\xi = 1$ — or any other gauge you like — that's OK as long as you use the same gauge for all the diagrams.

- ⋆ For extra credit, allow for an arbitrary gauge parameter ξ and work out how (the infinite parts of) all the counterterms depend on ξ.

3. Third, calculate the beta–functions for all the couplings of the theory to the one-loop order. After all the hard work you did in part (2), this should be simple.

4. Finally, consider the electromagnetic form factors $F_1(q^2)$ and $F_2(q^2)$ of the *neutral* Dirac field Ψ_n.

   (a) Draw *all* Feynman diagrams contributing to these form factors at the one-loop level. Do any of the counterterms of the renormalized perturbation theory contribute at this level? Explain your answer.

   (b) Evaluate the diagrams: introduce the Feynman parameters, reorganize the numerators according to the form factors F_1 and F_2, and integrate over the loop momentum. Write down formulae for the $F_1(q^2)$ and $F_2(q^2)$ in terms of integrals over the Feynman parameters. At this stage, allow for generic masses m_n, m_c and m_s and any q^2.

   (c) Verify that the F_1 form factor vanishes for $q^2 = 0$ but show that it does not vanish for generic $q^2 \neq 0$. This means that the radiative corrections give the neutral fermion a non-trivial electric charge density but the net charge remains zero.

   (d) Finally, calculate the net magnetic dipole moment of the neutral fermion. For simplicity, assume $m_s \approx m_c \gg m_n$.