Textbook question **Q14** (chapter 3):

Like all bodies moving under influence of gravity but no other forces, the rock in question has a downward acceleration. As long as the rock is flying up, it decelerates — its speed decreases. Consequently, it takes more time to move through the last 5 meters before the top than to move through the first five meters.

Textbook question **Q17** (chapter 3):

As long as the only important force acting on a body is gravity, it has a constant downward acceleration g. Thus, from the moment the ball is thrown till the moment it hits the ground, the ball has a downward acceleration g. This acceleration remains constant regardless of the direction of the ball's velocity: $a = -g$ points down while the ball flies up, $a = -g$ when the ball if falling down, and *at the top of the ball's flight, its acceleration is the same* $a = -g$*, where the minus sign indicates the downward direction.*

Non-textbook problem #**I**:

The ball goes up and down with constant downward acceleration $a = -g$, thus

$$v(t) = v_0 - gt, \tag{1}$$

$$y(t) = y_0 + v_0 t - \tfrac{1}{2} g t^2. \tag{2}$$

(a) The ball starts with positive (upward) initial velocity $v_0 = +15$ m/s. As the ball flies up, its velocity decreases according to eq. (1). When this velocity drops to zero, the ball reaches its highest point — since after that, the velocity turns negative and the ball starts falling down. Thus, the ball reaches the highest point when

$$v(t) = v_0 - gt = 0. \tag{3}$$

Solving this equation for the time t, we find

$$t = \frac{v_0}{g} = \frac{15\ \text{m/s}}{1\underline{0}\ \text{m/s}^2} = 1.5\ \text{s}. \tag{4}$$

(b) Given the time (4) when the ball reaches its highest point, the height of this point follows from eq. (2) evaluated for that time:

$$\begin{aligned} y_{\text{max}} &= y_0 + v_0 t_{\text{max}} - \tfrac{1}{2} g t_{\text{max}}^2 \\ &= 0 + (15\text{ m/s}) \times (1.5\text{ s}) - \tfrac{1}{2}(1\underline{0}\text{ m/s}^2)(1.5\text{ s})^2 \\ &= 0 + 22.5\text{ m} - 11.25\text{ m} \\ &= 11.25\text{ m} \approx 11\text{ m}. \end{aligned} \tag{5}$$

(c) There are two ways to solve this part of the problem. Given the maximal height (5) reached by the ball, the time it takes the ball to get back to the ground is the same as if the ball simply falls down from that height with no initial velocity:

$$t_{\text{fall}} = \sqrt{\frac{2y_{\text{max}}}{g}} = \sqrt{\frac{2(11.25\text{ m})}{1\underline{0}\text{ m/s}^2}} = \sqrt{2.25\text{ s}^2} = 1.5\text{ s}. \tag{6}$$

As I have explained in class, this falling time is equal to the time (4) the ball took to get up to the top from the ground.

Note that the time interval (6) begins when the ball is at its maximal height. The total flight time of the ball (up and down) is

$$t_{\text{net}} = t_{\text{rise}} + t_{\text{fall}} = 1.5\text{ s} + 1.5\text{ s} = 3.0\text{ s}. \tag{7}$$

Alternatively, we can find the net time of flight directly from the initial velocity v_0 and initial height $y_0 = 0$ by looking at the equation of motion (2) and solving for t such that $y(t) = 0$. That is, we need to solve the quadratic equation

$$y(t) = y_0 + v_0 t - \tfrac{1}{2} g t^2 = 0 \tag{8}$$

for the time t. Since the initial height y_0 is zero, this equation factorizes as

$$t \times (v_0 - \tfrac{1}{2} g t) = 0, \tag{9}$$

so the solutions are rather simple: $t = 0$ (when the motion starts with $y(0) = 0$) and

$$t = \frac{v_0}{\frac{1}{2}d} = \frac{2v_0}{g} = \frac{2(15 \text{ m/s})}{1\underline{0} \text{ m/s}^2} = 3.0 \text{ s} \tag{10}$$

when the ball falls back on the ground. Out of these 3 second, the ball goes up for 1.5 s (see part (a)), so for the remaining 3.0 s − 1.5 s = 1.5 s the ball falls down from the maximal height.

(d) The ball stays in the air for the total of 3.0 seconds. Its velocity at the end of this time follows from eq. (1):

$$v = v_0 - gt = (15 \text{ m/s}) - (1\underline{0} \text{ m/s}^2) \times (3.0 \text{ s}) = 15 \text{ m/s} - 3\underline{0} \text{ m/s} = -15 \text{ m/s}. \tag{11}$$

Note that this final velocity has the same magnitude as the initial velocity v_0 but opposite direction: down instead of up.

Non-textbook problem #**II**:

The flying bottle obeys the same equations of motion

$$v(t) = v_0 - gt, \tag{1}$$

$$y(t) = y_0 + v_0 t - \tfrac{1}{2}gt^2, \tag{2}$$

as the flying ball in the previous problem, only this time the initial position of the bottle is above the ground, $y_0 = 25$ m.

(a) A bottle hits the ground when its $y(t) = 0$. In light of eq. (2), this gives us a quadratic equation

$$y_0 + v_0 t - \tfrac{1}{2}gt^2 = 0 \tag{12}$$

for the time t. This equation has two solutions

$$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4(-g/2)y_0}}{2(-g/2)} = \frac{v_0 \pm \sqrt{v_0^2 + 2gy_0}}{g}, \tag{13}$$

but only the positive solution

$$t = \frac{v_0 + \sqrt{v_0^2 + 2gy_0}}{g} \tag{14}$$

corresponds to the time when the bottle hits the ground. The negative solution is unphysical since before $t = 0$ the bottle was not flying.

For the first bottle, the initial velocity is negative (downward) $v_0 = -2\underline{0}$ m/s. Consequently, the first bottle's time of flight is

$$t_1 = \frac{(-2\underline{0}\text{ m/s}) + \sqrt{(-2\underline{0}\text{m/s})^2 + 2(1\underline{0}\text{ m/s}^2) \times (25\text{ m})}}{1\underline{0}\text{ m/s}^2} \tag{15}$$

Under the square root here, we have

$$(-20\text{m/s})^2 + 2(10\text{ m/s}^2) \times (25\text{ m}) = 400\text{ m}^2/\text{s}^2 + 500\text{ m}^2/\text{s}^2 = 900\text{ m}^2/\text{s}^2 = (30\text{ m/s})^2, \tag{16}$$

hence the time of flight evaluates to

$$t_1 = \frac{-2\underline{0}\text{ m/s} + 3\underline{0}\text{ m/s}}{1\underline{0}\text{ m/s}^2} = 1.0\text{ s.} \tag{17}$$

(b) When the first bottle hits the ground, it has been flying for $t_1 = 1.0$ s. Hence, according to eq. (1), its final velocity is

$$v_1 = v_0^{(1)} - gt_1 = (-2\underline{0}\text{ m/s}) - (1\underline{0}\text{ m/s}^2) \times (1.0\text{ s}) = -3\underline{0}\text{ m/s.} \tag{18}$$

(c) Both bottles satisfy the same algebraic equation of motion (2), and solving it for the time of flight gives us the same general formula (14) for both bottles. However, the second bottle has a different initial velocity $v_0 = +2\underline{0}$ m/s, so its time of flight has a different numerical value

$$t_2 = \frac{(+2\underline{0}\text{ m/s}) + \sqrt{(+2\underline{0}\text{m/s})^2 + 2(1\underline{0}\text{ m/s}^2) \times (25\text{ m})}}{1\underline{0}\text{ m/s}^2} = \frac{+2\underline{0}\text{ m/s} + 3\underline{0}\text{ m/s}}{1\underline{0}\text{ m/s}^2} = 5.0\text{ s.} \tag{19}$$

(d) Similar to part (b), the final velocity of the second bottle is given by eq. (1). This time, the initial velocity is positive, but the time of flight is longer, thus the final velocity of the second bottle is

$$v_2 \;=\; v_0^{(2)} \;-\; gt_2 \;=\; +2\underline{0}\text{ m/s} \;-\; (1\underline{0}\text{ m/s}^2) \times (5.0\text{ s}) \;=\; -3\underline{0}\text{ m/s}. \tag{20}$$

Surprisingly, the second bottle hits the ground with the same final velocity as the first bottle.

(e) Actually, this is not so surprising since the equation

$$v^2 \;-\; v_0^2 \;=\; 2g(y_0 - y) \tag{21}$$

for the velocity of a free-falling body does not care for the sign of its initial velocity. Thus, when either bottle hits the ground, they both have the same value

$$2g(y_0 - y) \;=\; 2(1\underline{0}\text{m/s}^2) \times (25\text{ m} \;-\; 0\text{ m}) \;=\; 5\underline{0}0\text{ m}^2/\text{s}^2 \tag{22}$$

on the right hand side of eq. (21), so the left hand side of this equation should also be the same for both bottles. And indeed, both bottles have

$$v^2 \;-\; v_0^2 \;=\; (-3\underline{0}\text{ m/s})^2 \;-\; (\mp 2\underline{0}\text{ m/s})^2 \;=\; 9\underline{0}0\text{ m}^2/\text{s}^2 \;-\; 4\underline{0}0\text{ m}^2/\text{s}^2 \;=\; 5\underline{0}0\text{ m}^2/\text{s}^2. \tag{23}$$

PS: I have derived eq. (21) in class, but since it is not in the textbook, let me repeat the derivation here: First,

$$v^2 \;-\; v_0^2 \;=\; (v - v_0) \times (v + v_0) \;=\; (v - v_0) \times [2v_0 + (v - v_0)]. \tag{24}$$

Second, in light of eq. (1) for the velocity,

$$v \;-\; v_0 \;=\; -gt \quad\Longrightarrow\quad v^2 \;-\; v_0^2 \;=\; (-gt) \times [2v_0 - gt] \;=\; 2g \times \left(-v_0 t \;+\; \tfrac{1}{2}gt^2\right). \tag{25}$$

Finally, eq. (2) for the position leads to

$$y_0 \; - \; y \; = \; y_0 \; - \; \left(y_0 \, + \, v_0 t \; - \; \tfrac{1}{2} g t^2\right) = \; -v_0 t \; + \; \tfrac{1}{2} g t^2, \tag{26}$$

and the right hand sides of the last two equations, we immediately see that

$$v^2 \; - \; v_0^2 \; = \; 2g \times (y_0 - y). \tag{21}$$

Non-textbook problem #**III**:

Neglecting the height of the astronaut who threw the rock, we have the rock starting at height $y_0 = 0$ and eventually falling down to the same height $y_{\text{fin}} = 0$. Consequently, for the rock in question

$$t_{\text{up}} \; = \; t_{\text{down}} \; = \; \tfrac{1}{2} t_{\text{whole flight}} \; = \; \frac{22 \text{ s}}{2} \; = \; 11 \text{ s}. \tag{27}$$

The rock reaches its highest point at $t = t_{\text{up}}$; after that, it falls down just as if it was released with zero initial velocity at $t = t_{\text{up}}$. Consequently, the maximal height reached by the rock is the same as falling-down-from-rest distance during time $t_{\text{down}} = 11$ s,

$$y_{\max} \; = \; \tfrac{1}{2} g \times t_{\text{down}}^2 \, . \tag{28}$$

But since the rock flies up then falls down on the Moon, the g in this formula should be the Moon's gravity rather than the Earth's, $g = 1.6 \text{ m/s}^2$. Thus,

$$y_{\max} \; = \; \tfrac{1}{2}(1.6 \text{ m/s}^2) \times (11 \text{ s})^2 \; \approx \; 97 \text{ m} \; \approx \; 320 \text{ feet}. \tag{29}$$