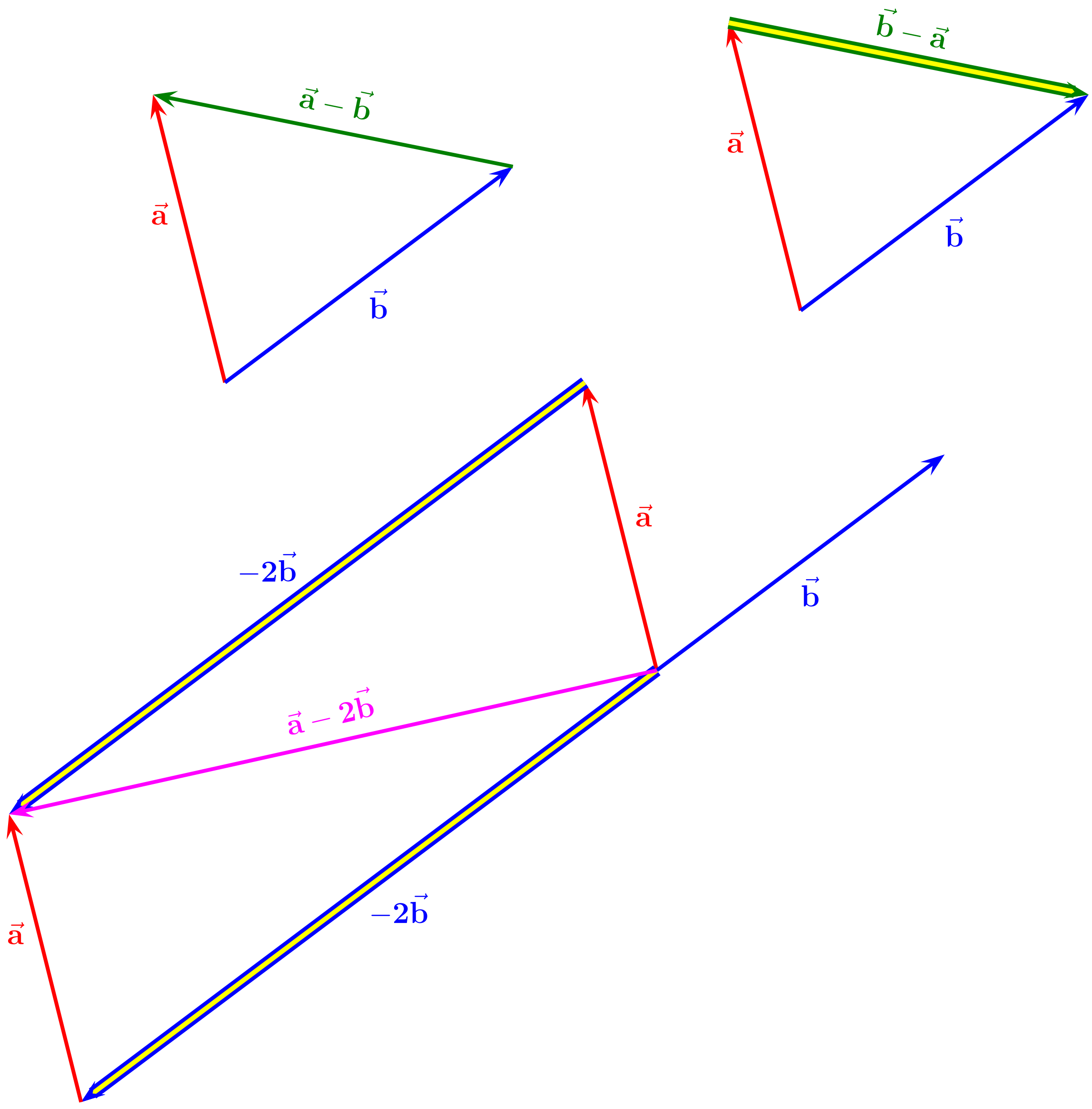
Non-textbook problem #**I**:



Non-textbook problem #**II**:

Since the yacht has constant velocity vector $\vec{\mathbf{v}}$, its displacement vector is simply

$$\Delta\vec{\mathbf{R}} \;=\; t\vec{\mathbf{v}}\,. \tag{1}$$

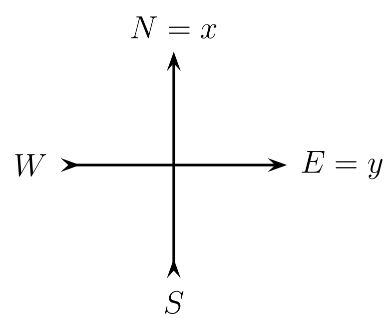
Since the time t is positive, this displacement vector has the same direction as the velocity

— namely SSW, 202.5° clockwise from North — while its magnitude is

$$|\Delta\vec{\mathbf{R}}| \;=\; t\times|\vec{\mathbf{v}}| \;=\; (6.0\text{ hr})\times(10.0\text{ knots}\equiv 10.0\text{ NM/hr}) \;=\; 6\underline{0}\text{ NM}, \tag{2}$$

(NM here denotes a nautical mile, 1 NM = 1852 m ≈ 6076 ft).

Now let's convert this displacement vector from magnitude and direction into components. In navigation, the direction angles are measured clockwise from due North (instead of counterclockwise from the x axis, whatever it is), so instead of the x and y components we should use the North and East components. This is equivalent to making the x axis point due North while the y axis points due East (instead of the usual direction of the axes on the graph),



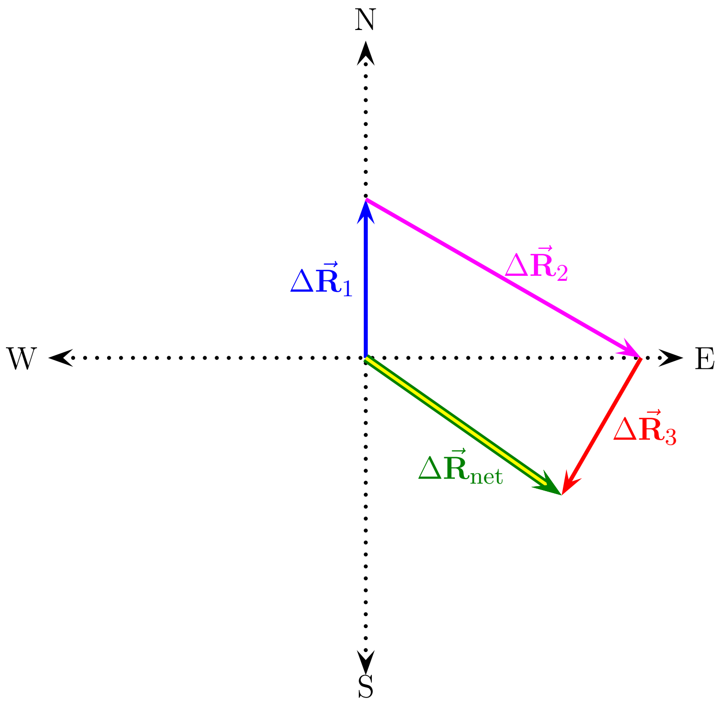
(3)

In this coordinate system,

$$\begin{aligned}
\Delta x \;\equiv\; \Delta R_N \;&=\; |\Delta\vec{\mathbf{R}}|\times\cos\theta,\\
&=\; 6\underline{0}\text{ NM}\times\cos(202.5^\circ) \;=\; -55\text{ NM},\\
\Delta y \;\equiv\; \Delta R_E \;&=\; |\Delta\vec{\mathbf{R}}|\times\sin\theta.\\
&=\; 6\underline{0}\text{ NM}\times\sin(202.5^\circ) \;=\; -23\text{ NM}.
\end{aligned} \tag{4}$$

Physically, the negative Northward component $\Delta R_N = -55$ NM means *Southward* displacement by 55 NM. Likewise, the negative Eastward component $\Delta R_E = -23$ NM means *Westward* displacement by 23 NM.

Non-textbook problem #**III**: (a)



(b) Similar to problem II, let us use the North and East components of vectors instead of x and y. Consequently, for any displacement vector $\Delta\vec{\mathbf{R}}$,

$$\Delta R_N \;=\; |\Delta\vec{\mathbf{R}}| \times \cos\theta, \qquad \Delta R_E \;=\; |\Delta\vec{\mathbf{R}}| \times \sin\theta, \tag{5}$$

where θ is the direction of $\Delta\vec{\mathbf{R}}$ according to the navigational rules, *i.e.* the clockwise angle from due North to the direction of $\Delta\vec{\mathbf{R}}$.

For the plane in question, we have 3 successive displacements with components

$$\begin{aligned} \Delta R_{1N} &= 90\text{ mi} \times \cos 0^\circ \;=\; +90\text{ mi}, \\ \Delta R_{1E} &= 90\text{ Mi} \times \sin 0^\circ \;=\; 0\text{ mi}, \\ \Delta R_{2N} &= 180\text{ mi} \times \cos 120^\circ \;=\; -90\text{ mi}, \\ \Delta R_{2E} &= 180\text{ mi} \times \sin 120^\circ \;=\; +156\text{ mi}, \\ \Delta R_{3N} &= 90\text{ mi} \times \cos 210^\circ \;=\; -78\text{ mi}, \\ \Delta R_{3E} &= 90\text{ mi} \times \sin 210^\circ \;=\; -45\text{ mi}. \end{aligned} \tag{6}$$

The net displacement of the plane is the vector sum of three displacements,

$$\Delta\vec{\mathbf{R}}_{\rm net} \;=\; \Delta\vec{\mathbf{R}}_1 \;+\; \Delta\vec{\mathbf{R}}_2 \;+\; \Delta\vec{\mathbf{R}}_3\,, \tag{7}$$

or in components,

$$\begin{aligned} \Delta R_N^{\rm net} \;&=\; \Delta R_{1N} \;+\; \Delta R_{2N} \;+\; \Delta R_{3N} \\ &=\; +90 \text{ mi} \;-\; 90 \text{ mi} \;-\; 78 \text{ mi} \;=\; -78 \text{ mi}, \\ \Delta R_E^{\rm net} \;&=\; \Delta R_{1E} \;+\; \Delta R_{2E} \;+\; \Delta R_{3E} \\ &=\; 0 \text{ mi} \;+\; 156 \text{ mi} \;-\; 45 \text{ mi} \;=\; +111 \text{ mi}. \end{aligned} \tag{8}$$

In other words, the net displacement of the plane is 78 miles southward (because $\delta R_N^{\rm net} < 0$) and 111 miles eastward.

(c) To obtain the magnitude of the net displacement vector $\Delta\vec{\mathbf{R}}_{\rm net}$ from the components (8), we use the Pythagoras's theorem

$$\Delta\vec{\mathbf{R}}_{\rm net}|^2 \;=\; \bigl(\Delta R_N{\rm net}\bigr)^2 \;+\; \bigl(\Delta R_E{\rm net}\bigr)^2 \;=\; (-78 \text{ mi})^2 \;+\; (111 \text{ mi})^2 \;\approx\; 18400 \text{ mi}^2 \tag{9}$$

and hence $|\Delta\vec{\mathbf{R}}_{\rm net}| = \sqrt{18400}$ mi ≈ 136 miles.

As to the direction θ of this net displacement, we have

$$\tan\theta \;=\; \frac{\delta R_E^{\rm net}}{\Delta R_N^{\rm net}} \;=\; \frac{+111 \text{ mi}}{-78 \text{ mi}} \;\approx\; -1.42, \tag{10}$$

and consequently

$$\theta \;=\; \arctan(-1.42) \;\approx\; -55^\circ \quad \textbf{modulo } 180^\circ. \tag{11}$$

That is, either $\theta = -55^\circ = +305^\circ$ or $\theta = -55^\circ + 180^\circ = 125^\circ$. To choose between these two possibilities, we notice that $\Delta R_N^{\rm net} < 0$ while $\Delta R_E^{\rm net} > 0$, which requires $\cos\theta < 0$ while $\sin\theta > 0$. Consequently, θ should lie somewhere between 90° and 180°, and indeed, looking at the diagram in part (a) we see that the direction of the net displacement lies somewhere in the southeastern quadrant, between 90° (due East) and 180° (due South). Consequently, the correct value of θ is 125° rather than $-55^\circ = +305^\circ$.

Non-textbook problem #**IV**:

Once the bag is dropped, it flies like a projectile with initial velocity vector $\vec{\mathbf{v}}_0$:

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}_0 + \vec{\mathbf{g}}t, \qquad \vec{\mathbf{R}}(r) = \vec{\mathbf{R}}_0 + \vec{\mathbf{v}}_0 t + \tfrac{1}{2}\vec{\mathbf{g}}t^2, \tag{12}$$

or in components,

$$\begin{aligned} v_x(t) &= v_{x0} = \text{const}, \\ v_y(t) &= v_{y0} - gt, \\ x(t) &= x_0 + v_{x0}t, \\ y(t) &= y_0 + v_{y0}t - \tfrac{1}{2}gt^2. \end{aligned} \tag{13}$$

(a) For the bag in question $v_{0y} = 0$, so its vertical motion is simply

$$y(t) = y_0 - \tfrac{1}{2}gt^2. \tag{14}$$

The bag hits the ground when this $y(t)$ drops to zero, so to find the time the bag was falling, we simply solve the equation $y(t) = 0$ for the time t. In light of eq.(14), we have

$$y_0 - \tfrac{1}{2}gt^2 = 0 \quad\Longrightarrow\quad \frac{g}{2} \times t^2 = y_0 \quad\Longrightarrow\quad t^2 = \frac{2y_0}{g} \tag{15}$$

and hence

$$t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2(180\text{ m})}{1\underline{0}\text{ m/s}^2}} \approx 6.0\text{ s}. \tag{16}$$

(b) While the bag is falling down, it is also moving horizontally according to the third equation (13). Its net horizontal displacement during the fall is

$$x(t) - x_0 = v_{0x}t = (45\text{ m/s}) \times (6.0\text{ s}) = 3\underline{0}0\text{ m}, \tag{17}$$

or about 1000 feet.

(c) According to the first two equations (13) evaluated for $t = 6.0$ s (when the bag hit the ground),

$$\begin{aligned} v_x &= v_{0x} = 45 \text{ m/s}, \\ v_y &= v_{0y} - gt = 0 - (1\underline{0} \text{ m/s}^2)(6.0 \text{ s}) = -6\underline{0} \text{ m/s}. \end{aligned} \tag{18}$$

(d) Converting the components (18) of the impact velocity vector into the magnitude — the *speed* of impact — and the direction, we have

$$v \equiv |\vec{\mathbf{v}}| = \sqrt{v_x^2 + v_y^2} = \sqrt{(45 \text{ m/s})^2 + (-60 \text{ m/s})^2} \approx 75 \text{ m/s}, \tag{19}$$

or about 170 MPH.

Also, the angle θ between the horizontal x axis and the impact velocity vector is

$$\theta = \arctan \frac{v_y = -60 \text{ m/s}}{v_x = +45 \text{ m/s}} = \arctan(-1.33) \approx -53^\circ. \tag{20}$$

The minus sign here indicates that the falling bag moves in the direction 53° *below* the horizontal.

<u>Non-textbook problem #**V**:</u>

First, consider a single water jet. Once a water droplet emerges from the nozzle, it flies like projectiles (since there are almost no forces between different droplets). Consequently, the whole jet follows the parabolic trajectory of a single droplet

$$y(x) = y_0 + \frac{v_{0y}}{v_{0x}} \times x - \frac{g}{2v_{0x}^2} \times x^2 \tag{21}$$

and hits the ground at the point x for which $y(x) = 0$. For a jet starting with zero initial elevation $y_0 = 0$, this point is given by the *horizontal range equation*

$$x = \frac{2v_{ox}v_{0y}}{g} = \frac{v_0^2}{g} \times 2\cos\theta_0 \sin\theta_0 = \frac{v_0^2}{g} \times \sin(2\theta). \tag{22}$$

(a) Now let's focus on the first jet in the fountain. For this jet we know the initial angle $\theta = \theta_1 = 15°$ but we do not know the initial velocity v_0. On the other hand, we know the horizontal range $x = 5$ m of this jet, so we can *solve the range equation for the v_0:*

$$\frac{v_0^2}{g} \times \sin(2\theta) \;=\; x \quad\Longrightarrow\quad v_0^2 \;=\; \frac{gx}{\sin(2\theta)} \;=\; \frac{(10\ \mathrm{m/s^2}) \times (5\ \mathrm{m})}{\sin(2 \times 15°)} \;=\; 100\ \mathrm{m^2/s^2} \qquad (23)$$

and hence the initial velocity is $v_0 = 10$ m/s.

(b) Finally, let's compare the two jets. They both start at the same point $(x_0 = 0, y_0 = 0)$ and hit the ground at the same point $(x = 5\ \mathrm{m}, y = 0)$, so applying the range equation (22) to each jet, we have

$$\frac{(v_0^{(1)})^2}{g} \times \sin(2\theta_1) \;=\; x \;=\; \frac{(v_0^{(2)})^2}{g} \times \sin(2\theta_2). \qquad (24)$$

Moreover, both jets have the same initial speed of the water, $v_0^{(1)} = v_0^{(2)} = v_0$, so both sides of the equation (24) have the same v_0^2/g factor. Dividing by this factor, we arrive at

$$\sin(2\theta_1) \;=\; \sin(2\theta_2), \qquad (25)$$

which naively seems to imply equal angles $\theta_1 = \theta_2$.

However, equal sines imply equal angles *only* when both angles are between 0° and 90°, but in the range equation (22) the sine is of the angle 2θ which ranges from 0° and $2 \times 90° = 180°$. In this range, two angles have the same sine, namely

$$\sin(\phi) \;=\; \sin(180° - \phi), \qquad (26)$$

or in terms of the $\phi = 2\theta$,

$$\sin(2\theta) \;=\; \sin(2(90° - \theta)) \qquad (27)$$

Therefore, eq. (25) for the two water jets implies

$$\textbf{either}\ \theta_2 \;=\; \theta_1\ \text{or}\ \theta_2 \;=\; 90° \;-\; \theta_1\,. \qquad (28)$$

Since we know that the first jet follows a low trajectory while the second jet follows a high trajectory, we have

$$\theta_2 \;=\; 90^\circ \;-\; \theta_1 \;=\; 90^\circ \;-\; 15^\circ \;=\; 75^\circ. \tag{29}$$

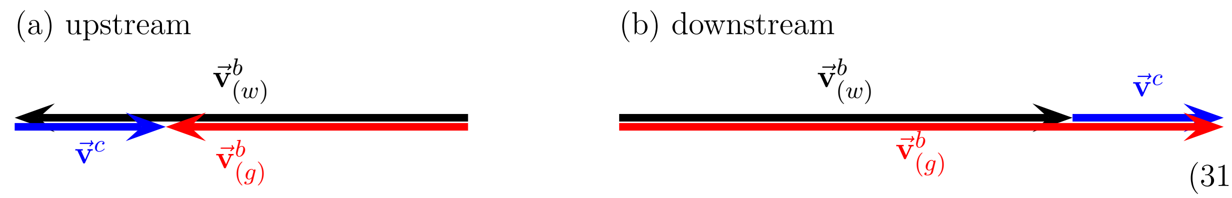
<u>Non-textbook problem #**VI**</u>:

Relative to the ground, the distance between the two cities is always the same 30 miles, regardless of the current or the boat's motion. So to find the time the boat needs to get from one city to the other, we need to know its speed relative to the ground. However, we are given the boat's speed relative to the water rather than to the ground, so this problem is about relating velocities of the same boat in two different frames of reference.

The general relation looks very simple in vector notations: The velocity $\vec{\mathbf{v}}^b_{(g)}$ of the boat relative to the ground is the *vector sum* of its velocity $\vec{\mathbf{v}}^b_{(w)}$ relative to the water and the current velocity $\vec{\mathbf{v}}^c$,

$$\vec{\mathbf{v}}^b_{(g)} \;=\; \vec{\mathbf{v}}^b_{(w)} \;+\; \vec{\mathbf{v}}^c. \tag{30}$$

The magnitude of this vector sum depends not only on the magnitudes of the $\vec{\mathbf{v}}^b_{(w)}$ and $\vec{\mathbf{v}}^c$ but also on the angle between them:



(31)

(**b**) When the boat steams downstream, the vectors $\vec{\mathbf{v}}^b_{(w)}$ and $\vec{\mathbf{v}}^c$ are in the same direction, so their magnitudes add up:

$$|\vec{\mathbf{v}}^b_{(g)} + \vec{\mathbf{v}}^c| \;=\; |\vec{\mathbf{v}}^b_{(w)}| \;+\; |\vec{\mathbf{v}}^c| \;=\; 15\text{ MPH} \;+\; 5\text{ MPH} \;=\; 20\text{ MPH}.$$

Thus, boat's speed *relative to the dry ground* is 20 MPH, and the time it takes to steam 30 miles downstream is

$$t_{\text{down}} \;=\; \frac{30\text{ miles}}{20\text{ miles/hr}} \;=\; 1.5\text{ hours}. \tag{32}$$

(**a**) When the boat steams upstream, the vectors $\vec{\mathbf{v}}^b_{(w)}$ and $\vec{\mathbf{v}}^c$ have opposite directions, so their magnitudes subtract instead of adding up:

$$|\vec{\mathbf{v}}^b_{(g)} + \vec{\mathbf{v}}^c| \;=\; |\vec{\mathbf{v}}^b_{(w)}| \;-\; |\vec{\mathbf{v}}^c| \;=\; 15\text{ MPH} \;-\; 5\text{ MPH} \;=\; 10\text{ MPH.}$$

Thus, boat's speed *relative to the dry ground* is 10 MPH, and the time it takes to steam 30 miles upstream is

$$t_{\text{up}} \;=\; \frac{30\text{ miles}}{10\text{ miles/hr}} \;=\; 3\text{ hours.} \tag{33}$$

Non-textbook problem #**VII**:

This is another relative motion problem: Given the velocity of the plane relative to the air, we want to find where the plane is going relative to the ground. Similar to the boat in the previous problem, the velocity vector $\vec{\mathbf{v}}^p_{(g)}$ of the plane relative to the ground is the vector sum of the plane's velocity vector $\vec{\mathbf{v}}^p_{(a)}$ relative to the air and the wind's velocity vector $\vec{\mathbf{v}}^w$,

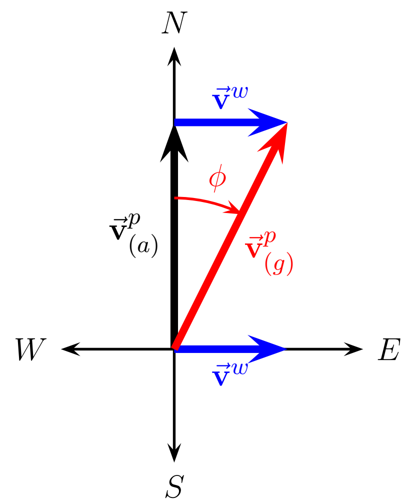
$$\vec{\mathbf{v}}^p_{(g)} \;=\; \vec{\mathbf{v}}^p_{(a)} \;+\; \vec{\mathbf{v}}^w. \tag{34}$$

But this time, the plane's motion is in two dimensions — North–South and East–West — so we need two equations for the N and E components of velocities:

$$\begin{aligned} v^p_{(g)N} \;&=\; v^p_{(a)N} \;+\; v^w_N\,, \\ v^p_{(g)E} \;&=\; v^p_{(a)E} \;+\; v^w_E\,. \end{aligned} \tag{35}$$

(a) In the first half of the problem, the plane *heads* due North while the wind blows from

West to East:



(36)

In (N, E) components, the plane's velocity vector relative to the air has

$$v^p_{(a)N} \;=\; +120 \text{ MPH}, \qquad v^p_{(a)E} \;=\; 0, \tag{37}$$

while the wind velocity vector has

$$v^w_N \;=\; 0, \qquad v^w_E \;=\; +60 \text{ MPH}. \tag{38}$$

Consequently, the plane's velocity vector relative to the ground has components

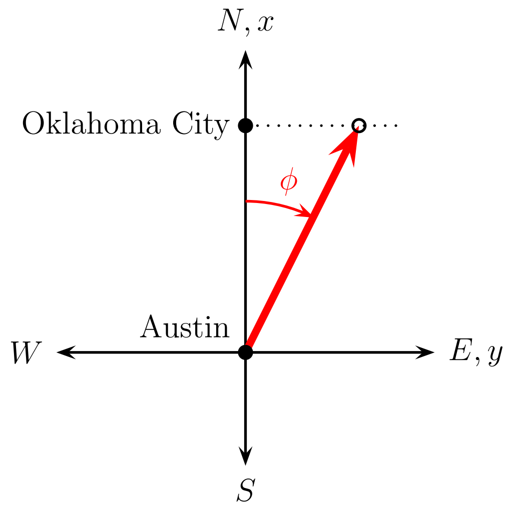
$$\begin{aligned} v^p_{(g)N} &= v^p_{(a)N} + v^w_N = +120 \text{ MPH} + 0 = +120 \text{ MPH}, \\ v^p_{(g)E} &= v^p_{(a)E} + v^w_E = 0 + +60 \text{ MPH} = +60 \text{ MPH}. \end{aligned} \tag{39}$$

The direction of this vector — *i.e.*, the direction of the plane's flight relative to the ground — is

$$\phi \;=\; \arctan\frac{v^p_{(g)y}}{v^p_{(g)x}} \;=\; \arctan\left(\frac{+60 \text{ MPH}}{+120 \text{ MPH}} = +0.5\right) \;\approx\; 27^\circ. \tag{40}$$

Thus, instead of flying due North to the Oklahoma City, the plane flies in the direction 27° (Eastward from due North).

(b) Here is the map of the plane's flight:

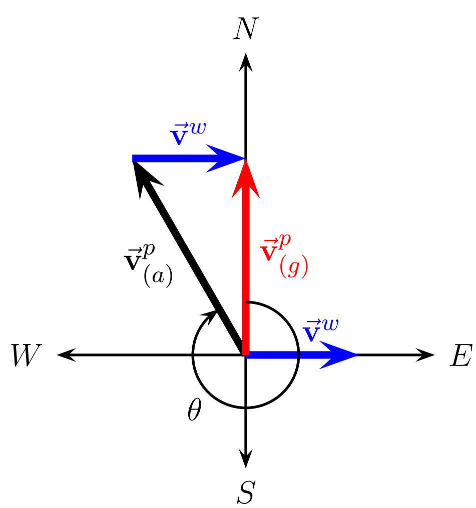


Oklahoma City is 360 miles due North from Austin, so in the coordinates of this picture it is at $x = +360$ miles, $y = 0$. By the time the plane flying in the direction $\phi = 27°$ reaches Oklahoma City's latitude — *i.e.*, $x = +360$ miles — it has moved in the y direction through

$$y = x \times \tan\phi = (360 \text{ miles}) \times (+0.5) = +180 \text{ miles}. \tag{41}$$

In other words, if the pilot does not know about the wind and *heads* due North, the plane would end up 180 miles East from Oklahoma City — in Fort Smith, Arkansas.

(c) In order to fly to Oklahoma City, the pilot should *head* upwind — *i.e.*, West from North — so that the plane's velocity vector *relative to the ground* points due North:



(42)

In components, this means

$$v^p_{(g)E} \;=\; v^p_{(a)E} \;+\; v^w_E \;=\; 0 \tag{43}$$

where

$$v^p_{(a)E} \;=\; |v^p_{(a)}| \times \sin\theta \;<\; 0, \tag{44}$$

$|v^p_{(a)}| = 120$ MPH is the known *airspeed* of the plane, and θ is the plane's heading (clockwise from due North, so expect $20^\circ < \theta < 360^\circ$). Combining the last two two equations, we obtain

$$|v^p_{(a)}| \times \sin\theta \;=\; -v^w_E\,, \tag{45}$$

hence

$$\sin\theta \;=\; -\frac{v^w_E}{|v^p_{(a)}|} = \;-\frac{+60\text{ MPH}}{120\text{ MPH}} \;=\; -0.5 \tag{46}$$

and therefore

$$\theta \;=\; \arcsin(-0.5) \;=\; -30^\circ \;\equiv\; +330^\circ. \tag{47}$$

Thus, to fly due North to Oklahoma City despite the wind from the West, the pilot should head the plane in the direction 330° — *i.e.*, 30° Westward from due North.

(d) To find the time of flight, we need the groundspeed of the plane. Since the direction of flight is due North, the groundspeed equals to the N component of the velocity relative to the ground,

$$|v^p_{(g)}| \;=\; v^p_{(g)N}\,, \tag{48}$$

which in turn can be obtained from the N components of the velocity relative to the air and of the wind's velocity,

$$v^p_{(g)N} \;=\; v^p_{(a)N} \;+\; v^w_N\,. \tag{49}$$

In fact, since the wind blows from West to East, the N component of its velocity is zero, $v^w_N = 0$, while the N component of the plane's velocity relative to the air follows from its known airspeed and heading,

$$v^p_{(a)N} \;=\; |\vec{\mathbf{v}}^p_{(a)}| \times \cos\theta \;=\; 120\text{ MPH} \times \cos(-30^\circ) \;\approx\; 104\text{ MPH}. \tag{50}$$

Hence, the N component of the ground velocity is

$$v^p_{(g)N} \;=\; 104\text{ MPH} \;+\; 0 \;=\; 104\text{ MPH}, \tag{51}$$

while the N component of the ground velocity is zero – the plane flies due North, that's why the pilot set the heading to 330°, (part (c)). Thus, the *groundspeed* of the plane is 104 MPH, and the time it takes to fly 360 miles from Austin to Oklahoma City is

$$t \;=\; \frac{L}{|v^p_{(g)}} \;=\; \frac{360\text{ miles}}{104\text{ miles/hr}} \;\approx\; 3.5\text{ hours}. \tag{52}$$

Note: If there were no wind, the plane's groundspeed would be the same as its airspeed — 120 MPH, — so it would get from Austin to Oklahoma City in just 3 hours. The wind reduces the groundspeed and makes the flight longer by half an hour.