Textbook question **Q25** (chapter **5**):

All three masses shown in diagram Q25 (textbook page 99) attract each other. In particular, m_1 attract m_2 with the force

$$F_{12} = \frac{Gm_1 m_2}{d_{12}^2} \qquad (1)$$

where d_{12} is the distance between masses m_1 and m_2. Likewise, m_3 attract m_2 with the force

$$F_{32} = \frac{Gm_1 m_2}{d_{32}^2}. \qquad (2)$$

The two forces pull m_2 in opposite directions: F_{12} pulls to the left — towards the m_1 — while F_{32} pulls to the right, towards the m_3. Let the x axis point to the right, then the x components of the two forces acting on the m_2 are

$$F_{12}^x = -\frac{Gm_1 m_2}{d_{12}^2} \quad \text{while} \quad F_{32}^x = +\frac{Gm_3 m_2}{d_{32}^2}. \qquad (3)$$

Altogether, the net force on the m_2 is

$$F_{\text{net on }2}^x = F_{12}^x + F_{32}^x = -\frac{Gm_1 m_2}{d_{12}^2} + \frac{Gm_3 m_2}{d_{32}^2} = Gm_2 \times \left(\frac{m_3}{d_{32}^2} - \frac{m_1}{d_1 2^2}\right). \qquad (4)$$

The problem specifies equal masses m_1 and m_3, while the diagram Q25 shows m_1 to be closer to m_2 than m_3, thus $d_{12} < d_{32}$. Consequently,

$$\frac{m_1}{d_{12}^2} > \frac{m_3}{d_{32}^2} \implies Gm_2 \times \left(\frac{m_3}{d_{32}^2} - \frac{m_1}{d_1 2^2}\right) < 0, \qquad (5)$$

so according to eq. (4)

$$F_{\text{net on }2}^x < 0. \qquad (6)$$

In other words, *the net force on the m_2 mass pulls it to the left*.

Non-textbook problem #**I**:

(a-b) According to Kepler's First Law, planets go around the Sun in *elliptic orbits*, with the Sun being in one of the focal points of the ellipse. This is illustrated at the *hyperhysics* web page about Kepler Laws: The top picture there shows the Sun located on the major axis of the ellipse (whose length is $2a$) at the distance $e \times a$ from the middle of the axis, where e is the *eccentricity* of the ellipse. The closest point of the orbit to the Sun — the *perihelion* — is at one end of the major axis, and the most distant point — the *aphelion* — it at the other end of the major axis. Thus, at the perihelion, the distance between the planet and the Sun is

$$R_p \;=\; a \;-\; e \times a \;=\; (1-e) \times a, \tag{7}$$

and at the aphelion, the distance is

$$R_a \;=\; a \;+\; e \times a \;=\; (1+e) \times a. \tag{8}$$

Mercury's orbit has semi-major axis $a = 0.387$ au and eccentricity $e = 0.20$. Hence, at the perihelion, Mercury is only

$$R_p \;=\; (1-0.20) \times 0.387\text{ au} \;\approx\; 0.31\text{ au} \;=\; 46 \cdot 10^6\text{ km}, \tag{9}$$

from the Sun, while at the aphelion the distance increases to

$$R_a \;=\; (1+0.20) \times 0.387\text{ au} \;\approx\; 0.46\text{ au} \;=\; 69 \cdot 10^6\text{ km}. \tag{10}$$

(c) Kepler's Third Law relates orbital periods of two planets orbiting the same star (or two satellites orbiting the same planet) to the average radii (semi-major axes) of their orbits:

$$\left(\frac{T_2}{T_1}\right)^2 \;=\; \left(\frac{a_2}{a_1}\right)^3 \qquad \textbf{or} \qquad \frac{T_2}{T_1} \;=\; \left(\frac{a_2}{a_1}\right)^{3/2}. \tag{11}$$

Mercury and Earth orbit the same Sun, so by this law

$$\frac{T_M}{T_E} \;=\; \left(\frac{a_M}{a_E}\right)^{3/2}. \tag{12}$$

By definition of the units, Earth's orbit has semi-major axis $a_E = 1$ au (exactly) and orbital period $T_E = 1$ year (exactly), so in those units, the period and the semi-major axis of

Mercury are related to each other as

$$T_M[\text{in Earth years}] \;=\; \big(a_M[\text{in au}]\big)^{3/2}. \tag{13}$$

The problem gives us $a_M = 0.387$ au, therefore

$$T_M[\text{in Earth years}] \;=\; (0.387)^{3/2} \;=\; 0.241. \tag{14}$$

In other words, Mercury's year is 0.241 Earth's years, or about 88 Earth's days.

Non-textbook problem **#II**:
According to Newton's Law of Gravity, your weight on Earth is

$$W_E \;=\; m \times g_E \;=\; \frac{G \times M_E \times m}{R_E^2} \tag{15}$$

where m is your mass, M_E and R_E are Earth's mass and radius, and G is the universal gravitational constant (also known as Newton's constant, even though Newton didn't known its value). Likewise, your weight on Titan would be

$$W_T \;=\; m \times g_T \;=\; \frac{G \times M_T \times m}{R_T^2}\,. \tag{16}$$

Note that G and m in eqs. (15) and (16) are exactly the same — G is universal, and your mass m does not depend on where you happen to be. Consequently, these variables cancel out of the *ratio*

$$\begin{aligned} \frac{W_T}{W_E} \;&=\; \frac{G \times M_T \times m}{R_T^2} \Big/ \frac{G \times M_E \times m}{R_E^2} \\ &=\; \frac{M_T}{R_T^2} \Big/ \frac{M_E}{R_E^2} \;=\; \frac{M_T/M_E}{(R_T/R_E)^2} \\ &=\; \frac{0.0226}{(0.468)^2} \;=\; 0.103. \end{aligned} \tag{17}$$

In other words, your weight on Titan would be 0.103 × your weight on Earth, whatever it happens to be. For example, if you weigh 150 pounds on Earth, on Titan you would weigh only 15.5 pounds.

Non-textbook problem #**III**:

The period T of a satellite's orbit around a planet follows from the orbit's semi-major axis a and the planet's (not the satellite's!) mass M as

$$T = 2\pi\sqrt{\frac{a^3}{GM}} \qquad (18)$$

where $G = 6.67428\cdot 10^{-11}\ \mathrm{N\cdot m^2/kg^2} = 6.67428\cdot 10^{-11}\ \mathrm{m^3/s^2/kg}$ is the gravitational constant.

In class, I have derived eq. (18) for circular orbits; let me remind you how it went. A satellite on a circular orbit of radius R and period T moves at constant speed

$$v = \frac{L = 2\pi R}{T}. \qquad (19)$$

Such motion has constant centripetal acceleration

$$a_c = \frac{v^2}{R} = \frac{(2\pi R/T)^2}{R} = \frac{(2\pi)^2 \times R}{T^2}, \qquad (20)$$

and the force ma_c providing this acceleration is the gravitational attraction of the planet,

$$ma_c = \frac{GMm}{R^2}. \qquad (21)$$

Combining the last two equations, we have

$$m \times \frac{(2\pi)^2 \times R}{T^2} = \frac{GMm}{R^2}, \qquad (22)$$

hence

$$\frac{(2\pi)^2}{T^2} = \frac{GM}{R^3} \qquad (23)$$

and therefore

$$\frac{T}{2\pi} = \sqrt{\frac{R^3}{GM}}. \qquad (24)$$

Calculating the period of a non-circular satellite orbit requires hard calculus, but the result is rather simple:

$$\frac{T}{2\pi} = \sqrt{\frac{a^3}{GM}}, \qquad (25)$$

similar to eq. (24) but using the semi-major axis a of the elliptic orbit instead of the circular orbit's radius R.

Given both the period T and the semi-major axis a of a satellite's orbit, we may solve eq. (25) for the planet's mass:

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^3}{GM}, \tag{26}$$

hence

$$M = \frac{(2\pi)^2 \times a^3}{G \times T^2}. \tag{27}$$

The orbit of Titan around Saturn has semi-major axis $a = 1.222 \cdot 10^9$ meters (about 760,000 miles, 3.16 times large than the Moon's orbit around the Earth) and period $T = 1\,377\,600$ seconds (about 16 days, 0.584 of the Moon's orbital period), so Saturn's mass comes out to be

$$\begin{aligned} M_S = \frac{(2\pi)^2 \times a_T^3}{G \times T_T^2} &= \frac{(2\pi)^2 \times (1.222 \cdot 10^9 \text{ m})^3}{(6.67428 \cdot 10^{-11} \text{ m}^3/\text{s}^2/\text{kg}) \times (1.3776 \cdot 10^6 \text{ s})^2} \\ &= \frac{72.04 \cdot 10^{27} \text{ m}^3}{126.66 \text{ m}^3/\text{kg}} = 5.688 \cdot 10^{26} \text{ kg}, \end{aligned} \tag{28}$$

about 95 Earth's masses.

Textbook question **Q2** (chapter **6**):

The mechanical work is a very different concept from the everyday meaning of person's work or labor. By definition,

$$\text{mechanical work} = \text{force} \times \text{displacement(in the direction of the force)}, \tag{29}$$

so unless you apply a force to a *moving* body, you are not performing any mechanical work, regardless how hard you might feel you are working.

The man in question is pushing very hard upon a rock, but the rock does not move, so its displacement is zero. Consequently, the mechanical work done by the man on the rock is zero, despite all of the man's efforts.

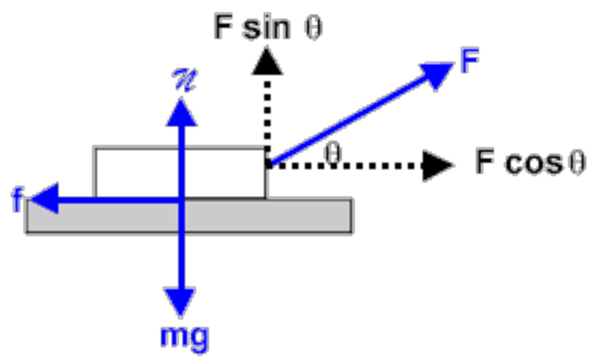
Non-textbook problem #**IV**: By definition,

$$\begin{aligned}\text{mechanical work} &= \text{force} \times \text{displacement(in the direction of the force)}\\ &= \text{displacement} \times \text{force(in the direction of the displacement)}\\ &= \text{displacement} \times \text{force} \times \cos\begin{pmatrix}\text{angle between the force and}\\ \text{the displacement vectors}\end{pmatrix},\end{aligned}$$

or in vector notations

$$\begin{aligned}W &= \vec{\mathbf{D}} \cdot \vec{\mathbf{F}}\\ &= |\vec{\mathbf{D}}| \times |\vec{\mathbf{F}}| \times \cos\bigl(\text{angle between } \vec{\mathbf{D}} \text{ and } \vec{\mathbf{F}}\bigr)\\ &= D_x \times F_x \;+\; D_y \times F_y \qquad\qquad [\text{in 2D}]\\ &= D_x \times F_x \;+\; D_y \times F_y \;+\; D_z \times F_z \quad [\text{in 3D}].\end{aligned} \tag{30}$$

Now consider the box in question and the forces acting on it:



The box moves horizontally (along the floor) to the right, so the mechanical work of a force follows from the force's horizontal component (along the displacement): If the horizontal component points to the right, its work is positive, if its directed left, the work is negative, and if a force is vertical and has no horizontal component, then the mechanical work is zero.

In particular: (1) The force $\vec{\mathbf{F}}$ of the pulling has a rightward horizontal component, so its work is positive. (2) The friction force $\vec{\mathbf{f}}$ pulls to the left, so its work is negative. (3) The gravity force $m\vec{\mathbf{g}}$ and the normal force $\vec{\mathbf{N}}$ from the floor are both vertical, so these forces perform zero work.

Non-textbook problem #**V**:

To maintain a constant speed of the bucket, the man should pull on the rope with force T equal to the full bucket's weight $mg = 10.0 \text{ kg} \times 9.8 \text{ m/s}^2 = 98$ N. This force acts on the bucket over a distance L equal to the well's depth D, and the direction of the force is the same as the direction of the bucket's displacement. Hence, the mechanical work is

$$W = T \times L = mg \times D. \tag{31}$$

Given the force and the work, we can use this equation to find the well's depth as

$$D = \frac{W}{mg} = \frac{5900 \text{ J}}{98 \text{ N}} \approx 6\underline{0} \text{ m}, \tag{32}$$

or about 200 feet.

Non-textbook problem #**VI**:

(a) There are four forces acting on the sleigh: its weight mg (directed down), the normal force N from the snow-covered ground (directed up) the kinetic friction force f (directed back) and the pulling force T from the horse (directed forward). To keep the sleigh moving at constant velocity, all these forces should cancel, thus

$$\begin{aligned} ma_x &= F_x^{\text{net}} = T - f = 0, \\ ma_y &= f_y^{\text{net}} = N - mg = 0, \end{aligned} \tag{33}$$

and consequently

$$N = mg, \quad T = f. \tag{34}$$

Also, the kinetic friction force f is controlled by the normal force N,

$$f = \mu_k \times N, \tag{35}$$

therefore

$$T = f = mu_k \times N = \mu_k \times mg. \tag{36}$$

Numerically, the horse should pull the sleight in question with force

$$T = \mu_k \times mg = 0.12 \times 160 \text{ kg} \times 9.8 \text{ N/kg} = 188 \text{ N} \approx 190 \text{ N}. \tag{37}$$

(b) The horse pulls the sleight with a force $\vec{\mathbf{T}}$ in the direction of the sleigh's velocity vector $\vec{\mathbf{v}}$, so the power produced by the horse is

$$P = \vec{\mathbf{T}} \cdot \vec{\mathbf{v}} = Tv. \tag{38}$$

Given the force T and the maximal power $P_{\max}$ the horse is capable of, we can find the maximal speed at which he can pull the sleigh as

$$P_{\max} = T \times v_{\max} \quad \Longrightarrow \quad v_{\max} = \frac{P_{\max}}{T}\,. \tag{39}$$

Numerically, for the sleigh in question,

$$v_{\max} = \frac{1.00\ \text{hp} = 746\ \text{W}}{188\ \text{N}} = 3.96\ \text{m/s} \approx 4.0\ \text{m/s} \tag{40}$$

or about 9 miles per hour.