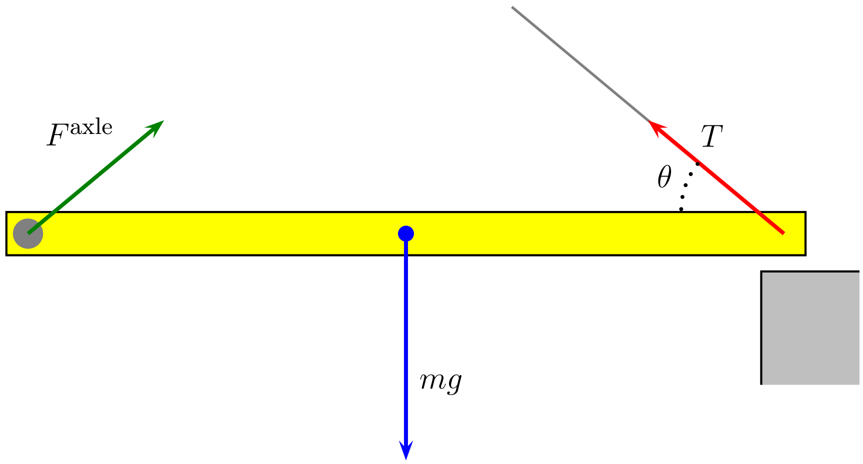
Non-textbook problem #**I**:

Let's start with a schematic side view of the drawbridge and the forces acting on it:



The bridge is shown just as it begins lifting, so it is almost horizontal but no longer in contact with the right support. Thus, there is no normal force on the right and of the bridge. Instead, the right end is pulled up by the two chains of net tension T; we do not know the magnitude of this tension force but we do know its direction: along the chains themselves, $\theta = 40°$ above the horizontal.

On its left end, the bridge is pivoted on an axle. The axle is well greased, so the bridge can swing up or down without frictional torque, but the axle itself cannot move horizontally of vertically. The axle exerts some force F^{axle} on the left end of the bridge, but we do not know the magnitude or the direction of this force.

Finally, the last force acting on the bridge is its own weight mg; its magnitude is 3200 N and its direction is straight down.

Now let's calculate the torques of all these forces relative to the axle. The axle force F^{axle} acts right at the axle, so regardless of the direction of the axle force it has zero lever arm and hence zero torque,

$$\tau\left(F^{\text{axle}}\right) \;=\; 0. \tag{1}$$

The chain tension T acts at the right end of the bridge, at distance $L = 3.3$ m for the axle, but its direction makes angle $\theta = 40^\circ$ with the bridge, so its lever arm is $L \times \sin\theta$ and the torque is

$$\tau(T) \;=\; +T \times L \times \sin\theta. \tag{2}$$

Finally, the bridge's own weight mg acts all over the bridge, but its torque is the same as if the whole weight was concentrated at the bridge's center of gravity — which is in the middle of the bridge, at distance $\frac{1}{2}L$ from the axle. Thus, the torque of the bridge's weight is

$$\tau(mg) \;=\; -mg \times \frac{L}{2}\,. \tag{3}$$

Altogether, the net torque on the bridge is

$$\tau^{\text{net}} \;=\; \tau\bigl(F^{\text{axle}}\bigr) \;+\; \tau(T) \;+\; \tau(mg) \;=\; 0 \;+\; T \times L\sin\theta \;-\; mg \times \frac{L}{2}\,. \tag{4}$$

To start lifting the bridge, we need to give it a small counterclockwise angular accleration $\alpha > 0$, thus we need a conterclockwise net torque

$$\tau^{\text{net}} \;=\; I \times \alpha \;>\; 0. \tag{5}$$

However, medieval drawbridges are lifted rather slowly, with a rather small the angular accleration α, so we may approximate $\alpha \approx 0$ — which calls for

$$\tau^{\text{net}} \;\approx\; 0. \tag{6}$$

In light of eq. (4), this means

$$T \times L\sin\theta \;-\; mg \times \frac{L}{2} \;\approx\; 0 \tag{7}$$

and hence

$$T \;\approx\; mg \times \frac{\frac{1}{2}L}{L\sin\theta} \;=\; \frac{mg}{2\sin\theta} \tag{8}$$

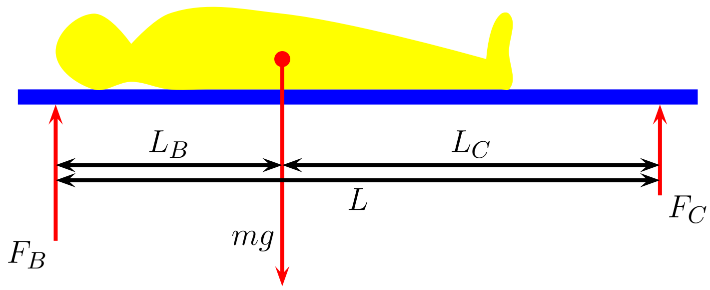
Note that the bridge's length L cancels out from this formula!

Numerically, for the bridge in question

$$T = \frac{mg}{2\sin\theta} = \frac{3200\text{ N}}{2\sin 40^\circ} \approx 2500\text{ N}, \tag{9}$$

about 560 pounds.

Non-textbook problem #**II**:



The equilibrium conditions for any rigid body are

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum \tau = 0. \tag{10}$$

The stretcher in question is subject to three forces: The patient's weight Mg, force F_B of Bob's hands, and force F_C from Charlie's hands. All these forces are vertical, so $\sum F_x = 0$ is trivially true, and there is only one non-trivial balance-of-forces equation,

$$\sum F_y = F_B + F_C - Mg = 0. \tag{11}$$

In the balance-of-torques equation $\sum \tau = 0$, we may treat any point P we like as a pivot, as long as we calculate toques of all forces with respect to the same pivot point P. For the problem at hand, it's convenient to put P at one end of the stretcher, for example at the front end where Bob holds the stretcher. For this choice of a pivot, the force F_B has zero lever arm and hence zero torque. On the other hand, the force F_C acts at the other end of the stretcher, so its lever arm is $L = 8$ ft — the full length of the stretcher — and the torque is $\tau(F_C) = +F_C \times L$, where the '+' sign indicates the counterclockwise direction of this torque.

Finally, the patient's weight Mg is distributed all over the patient's body, but for the purpose of calculating the torque we may treat it as acting at the patient's center of gravity (same as his center of mass). This center of mass lies at $L_B = 3$ ft from the pivot point (Bob's hands), so the lever arm of Mg is L_B and the torque is $\tau(Mg) = -Mg \times L_B$, where the '$-$' sign indicates the clockwise direction of this torque.

Altogether, the net torque around out chosen pivot point is

$$\sum \tau \;=\; \tau(F_B) \;+\; \tau(F_C) \;+\; \tau(Mg) \;=\; 0 \;+\; F_C \times L \;-\; Mg \times L_B\,. \tag{12}$$

Demanding that this net torque cancels out, we have

$$F_C \times L \;-\; Mg \times L_B \;=\; 0 \tag{13}$$

and therefore

$$F_C \;=\; Mg \times \frac{L_B}{L} \;=\; 200\text{ lb} \times \frac{3\text{ ft}}{8\text{ ft}} \;=\; 80\text{ lb.} \tag{14}$$

Consequently, according to the balance-of-forces equation (11),

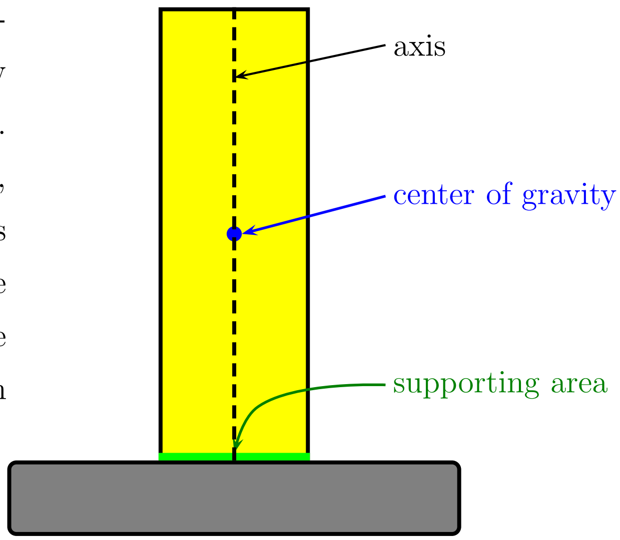
$$F_B \;=\; Mg - F_C \;=\; Mg - Mg\times\frac{L_B}{L} \;=\; Mg\times\left(1 \;-\; \frac{L_B}{L} \;=\; \frac{L_C}{L}\right) \;=\; 200\text{ lb}\times\frac{5\text{ ft}}{8\text{ ft}} \;=\; 120\text{ lb.} \tag{15}$$

Thus, Charlie carries 80 pounds of patient's weight and Bob carries the remaining 120 pounds.

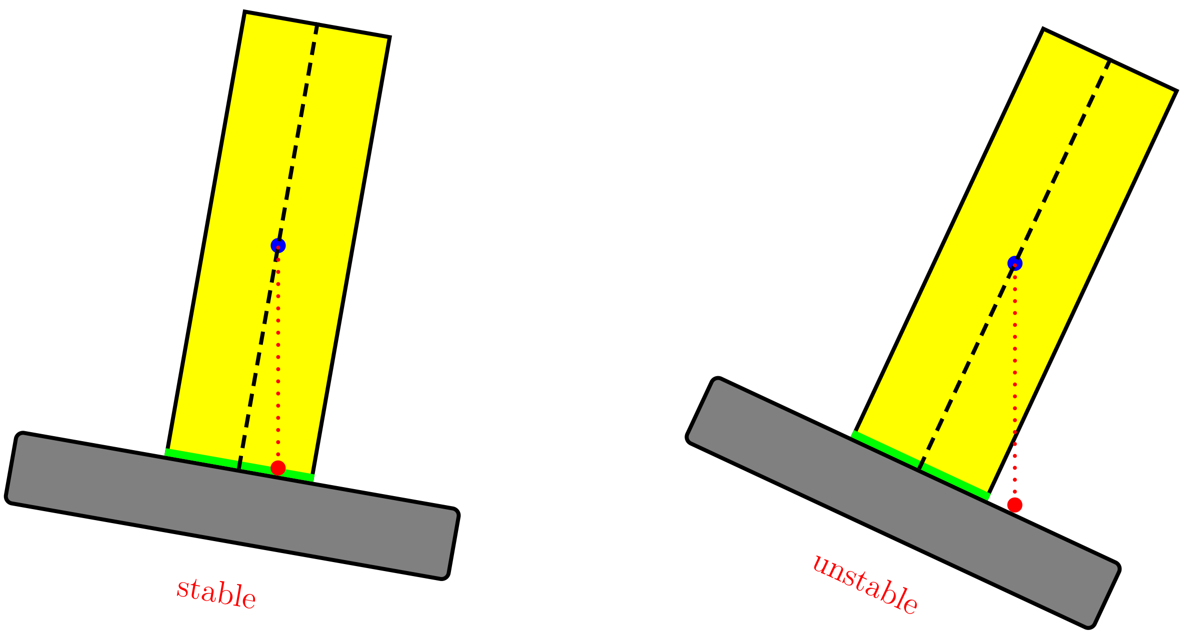
Non-textbook problem **#III**:

A standing body is stable as long as its center of gravity lies above its supporting area (the actual supports and anything between them), *i.e.*, if the vertical projection of the center of gravity down to the ground lies within the supporting area. If the center of gravity moves so that its projection gets outside the supporting area, the body topples to a side.

The body in question is a cylindrical column standing on its end — in the side view diagram at the right it's the yellow rectangle. The column has uniform diameter and density, so its center of gravity is on the cylinder's axis at $\frac{1}{2}$ of the column's height, *i.e.* 6 feet above the base platform. The supporting area of the column is its bottom end where it stands on the base; it's a circle, although in the side view diagram at the right only its diameter is visible (green horizontal line).

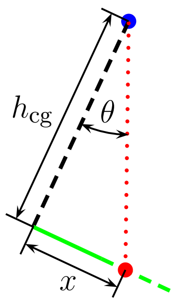


When the base platform is horizontal and the column is vertical, the center of gravity is directly above the center of the bottom circle. But when the base platform tilts and the column leans away from the vertical, the center of gravity moves to a side, so if you project it vertically down, the projection — denoted by the red dot on the diagrams below — would no longer be at the center of the base.



As long as the projection stays within the base — even if no longer at the center — the column would be stable, but if the projection moves outside the base, the column will topple.

To find how the projection moves as the column leans away from the vertical, we may use the right triangle at the left made by the column's axis and base (short sides) and the vertical line from the center of gravity down to its projection (the long side). On the diagram, θ is the angle by which the column leans away from the vertical; $h_{\rm cg} = 6$ feet is the height of the center of gravity back when the column was vertical, and also the distance between the CoG and the center of the base of a leaning column; and x is the distance by which CoG projection moves away from the base center. Solving this right triangle, we see that this distance is



$$x \;=\; h_{\rm cg} \times \tan\theta \tag{16}$$

Strictly speaking, x in eq. (16) is not the horizontal displacement but the displacement along the base of the column; the horizontal displacement is

$$x_{\rm horizontal} \;=\; x \times \cos\theta \;=\; h_{\rm cg} \times \sin\theta \tag{17}$$

In particular, when the column in questions was leaning 1° from the vertical, the horizontal displacement was (6 feet) $\times \sin 1^\circ \approx 1.25$ inch. (This is the answer to part (b).)

But for part (c) we need the displacement x along the base as in eq. Xcol. As long as x is less than the column radius $r = 0.25$ foot,[⋆] the CoG is inside the base and the column is stable in its off-vertical position, but when x becomes larger than the radius, the CoG is no longer above the base and the column topples. Thus,

$$\begin{aligned} \text{the column is stable when } x \;&=\; h_{\rm cg} \times \tan\theta \;<\; r, \\ \text{the column topples down when } x \;&=\; h_{\rm cg} \times \tan\theta \;>\; r. \end{aligned} \tag{18}$$

⋆ The column diameter $2r$ is 6 inches, *i.e.* 0.5 foot, so its radius is only 3 inches or 0.25 foot.

In terms of the angle θ, the column is stable while

$$\tan\theta \;<\; \frac{r}{h_{\rm cg}} \;=\; \frac{0.25\ \rm ft}{6\ \rm ft} \;=\; 0.042 \quad\Longrightarrow\quad \theta \;<\; \arctan\frac{r}{h_{\rm cg}} \;=\; 2.4^\circ. \tag{19}$$

When the column leans away from the vertical by more than 2.4°, it loses stability and topples down.

Problem #**IV**:

(a) By symmetry, the center of mass of a solid cylinder of uniform density lies on the cylinder axis, half way between the two ends of the cylinder. For a vertical cylinder, half-way means at one half of the cylinder's height. In particular, the 1.00 m long cylinder in question has its center of mass 0.50 m above the cylinder's bottom. Since the cylinder stands on the floor, its center of mass is 0.50 m above the floor,

$$Y_{\rm cm}[\text{cylinder}] \;=\; 0.50\ \text{m}. \tag{20}$$

(b) A hollow sphere of uniform thickness and density is symmetric with respect to rotations around any axis through the sphere's center, so its center of mass is at the geometric center of the sphere.

So let's locate the center of the bronze sphere in question. The sphere stands on top of a 1.00 m high cylinder standing on the floor, so the bottom of the sphere is 1.00 m above the floor. The geometric center of the sphere is higher then the bottom by the sphere's radius $R = 0.50$ m, so it lies 1.50 m above the floor. The sphere's center of mass is at the same location, thus

$$Y_{\rm cm}[\text{sphere}] \;=\; 1.50\ \text{m}. \tag{21}$$

(c) The sculpture consists of two parts, so once we find the centers of mass of each part, the center of mass of the whole sculpture is at

$$\vec{\mathbf{R}}_{\rm cm}[\text{whole}] \;=\; \frac{M_{\rm cyl}}{M_{\rm cyl}+M_{\rm sph}}\times\vec{\mathbf{R}}_{\rm cm}[\text{cylinder}] \;+\; \frac{M_{\rm sph}}{M_{\rm cyl}+M_{\rm sph}}\times\vec{\mathbf{R}}_{\rm cm}[\text{sphere}]. \qquad (22)$$

In particular, its vertical coordinate is at

$$Y_{\rm cm}[\text{whole}] \;=\; \frac{M_{\rm cyl}}{M_{\rm cyl}+M_{\rm sph}}\times Y_{\rm cm}[\text{cylinder}] \;+\; \frac{M_{\rm sph}}{M_{\rm cyl}+M_{\rm sph}}\times Y_{\rm cm}[\text{sphere}]. \qquad (23)$$

The problem gives us the masses of the two parts, while the heights of their respective center of mass were found in parts (a) and (b). Plugging all these data into eq. (23), we immediately obtain

$$Y_{\rm cm}[\text{whole}] \;=\; \frac{340\text{ kg}}{340\text{ kg}\,+\,160\text{ kg}}\times 0.50\text{ m} \;+\; \frac{160\text{ kg}}{340\text{ kg}\,+\,160\text{ kg}}\times 1.50\text{ m} \;=\; 0.82\text{ m}. \quad (24)$$

Non-textbook problem #**V**:

(a) The rotation rate is the angular velocity ω in non-metric units of rev/s (RPS) or rev/min (RPM). For the bicycle wheel in question,

$$\omega \;=\; 3000\text{ rev/min}\times\frac{1\text{ min}}{60\text{ s}}\times 2\pi\text{ rad/rev} \;=\; 314\text{ rad/s}. \qquad (25)$$

(b) The linear speed of a point at distance r from the axis of a rotating body is $v = r\times\omega$, provided ω is in radians/second. The nipple of the tire is at $r = 28$ cm $= 0.28$ m, so its speed is

$$v \;=\; r\times\omega \;=\; 0.28\text{ m}\times 314\text{ rad/s} \;=\; 88\text{ m/s}, \qquad (26)$$

almost 200 miles per hour.

(c) Assuming constant torque and hence constant angular acceleration, the angular velocity of the wheel changes with time as

$$\omega(t) \;=\; \omega_0 \;+\; \alpha t. \tag{27}$$

If the wheel stops after time t, then

$$\omega(t) \;=\; \omega_0 \;+\; \alpha t \;=\; 0 \quad\Longrightarrow\quad \alpha \;=\; -\frac{\omega_0}{t}\,. \tag{28}$$

For the wheel in question, the negative angular acceleration was

$$\alpha \;=\; -\frac{\omega_0}{t} \;=\; -\frac{314\text{ rad/s}}{30\text{ s}} \;=\; -10.5\text{ rad/s}^2. \tag{29}$$

(d) A body rotating at constant angular acceleration (or deceleration) rotates in time t through angle

$$\Delta\phi \;=\; \phi(t) \;-\; \phi_0 \;=\; \omega_0 t \;+\; \tfrac{1}{2}\alpha t^2. \tag{30}$$

For deceleration to stop, $\alpha = -\omega_0/t$ as in part (c), hence

$$\Delta\phi \;=\; \omega_0 t \;+\; \frac{-\omega_0}{t} \times \frac{t^2}{2} \;=\; \omega_0 t \;-\; \tfrac{1}{2}\omega_0 t \;=\; \tfrac{1}{2}\omega_0 t. \tag{31}$$

For the wheel in question,

$$\Delta\phi \;=\; \tfrac{1}{2}\omega_0 t \;=\; \tfrac{1}{2} \times 314\text{ rad/s} \times 30\text{ s} \;=\; 4712\text{ rad} \;=\; 750\text{ rev}, \tag{32}$$

the wheel makes 750 turns before stopping.

Non-textbook problem #**VI**:

(a) A disk of uniform density and thickness has moment of inertia

$$I \;=\; \frac{1}{2} \times M \times R^2. \tag{33}$$

For the grinding wheel in question, the moment of inertia is

$$I \;=\; \frac{1}{2} \times 2.4\text{ kg} \times (0.40\text{ m})^2 \;=\; 0.192\text{ kg}\cdot\text{m}^2 \;\approx\; 0.19\text{ kg}\cdot\text{m}^2. \tag{34}$$

(b) The kinetic energy of a rotating body is

$$K = \tfrac{1}{2}I\omega^2. \tag{35}$$

The moment of inertia I of the grinding wheel was found in part (a) while its angular velocity is

$$\omega = 720\text{ RPM} = 720\text{ rev/min} \times \frac{1\text{ min}}{60\text{ s}} \times 2\pi\text{ rad/rev} = 75.4\text{ rad/s} \approx 75\text{ rad/s}, \tag{36}$$

so its kinetic energy is

$$K = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2} \times 0.192\text{ kg}\cdot\text{m}^2 \times (75.4\text{ rad/s})^2 \approx 550\text{ J}. \tag{37}$$

(c) The angular momentum of a rotating body is

$$L = I \times \omega. \tag{38}$$

For the grinding wheel in question,

$$L = 0.192\text{ kg}\cdot\text{m}^2 \times 75.4\text{ rad/s} \approx 14.5\text{ N}\cdot\text{s}\times\text{m}. \tag{39}$$

(d) The angular momentum changes with time at the rate equal to the net torque on the body,

$$\Delta L = \tau^{\text{net}} \times \Delta t. \tag{40}$$

Once the motor is turned off, the angular momentum of the wheel changes from $L = 14.5\text{ N}\cdot\text{s}\times\text{m}$ to zero in $\Delta t = 40$ second, hence the net torque (without the motor) must have been

$$\tau^{\text{net}} = \frac{\Delta L}{\Delta t} = \frac{0 - 14.5\text{ N}\cdot\text{s}\times\text{m}}{40\text{ s}} \approx -0.36\text{ N}\times\text{m}. \tag{41}$$

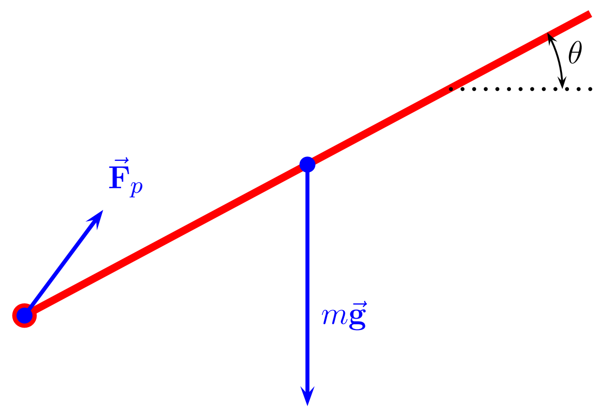
Without the motor, there is only the negative torque from the the friction forces on the grinding wheel, so this torque is $\tau = -0.36\text{ N}\times\text{m}$.

Non-textbook problem #**VII**:

The rod is pivoted around its free end, so assuming the rod has uniform thickness and density, its moment of inertia is

$$I = \frac{1}{3} \times M \times L^2 = \frac{1}{3} \times 1.20\ \text{kg} \times (1.00\ \text{m})^2 = 0.400\ \text{kg} \cdot \text{m}^2. \tag{42}$$

(b) There are two forces acting on the rod once it's released, its own weight $m\vec{\mathbf{g}}$ and the unknown force $\vec{\mathbf{F}}_p$ at the pivot:



The $\vec{\mathbf{F}}_p$ acts at the pivot, so regardless of its direction it has zero lever arm and hence zero torque. As to the gravity force, it is distributed all over the rod's length, but its torque is the same as if it was acting at the rod's center of gravity. Since rod is uniform, the center of gravity is in its middle, at distance $\frac{1}{2}L = 0.500$ m from the pivot. The lever arm of the mg force is the horizontal component of this distance,

$$\ell = \tfrac{1}{2}L \times \cos\theta = 0.500\ \text{m} \times \cos 30^\circ = 0.433\ \text{m}, \tag{43}$$

so the torque of the rod's weight is

$$\tau(mg) = mg \times \ell = mg \times \tfrac{1}{2}L\cos\theta = 1.20\ \text{kg} \times 9.8\ \text{N/kg} \times 0.433\ \text{m} = 5.09\ \text{N} \times \text{m}. \tag{44}$$

(c) By the rotational analogy of the Newton's Second Law,

$$I \times \alpha \;=\; \tau^{\rm net} \quad\Longrightarrow\quad \alpha \;=\; \frac{\tau^{\rm net}}{I}\,. \tag{45}$$

Since the pivot force $\vec{\mathbf{F}}_p$ does not have any torque, the net torque on the rod is the torque of its weight,

$$\tau^{\rm net} \;=\; \tau(mg) \;=\; mg \times \tfrac{1}{2}L\cos\theta \;=\; 5.09\ {\rm N}\times{\rm m}. \tag{46}$$

Consequently, the rod has angular acceleration

$$\alpha \;=\; \frac{\tau^{\rm net}}{I} \;=\; \frac{5.09\ {\rm N}\times{\rm m}}{0.400\ {\rm kg}\cdot{\rm m}^2} \;=\; 12.7\ {\rm rad/s}^2. \tag{47}$$

(d) At the moment the rod is released, it does not have any angular velocity yet, so the free end has zero linear speed and zero centripetal acceleration. But the angular acceleration of the rod gives its free end (which is at distance L from the axis) a *tangential* acceleration

$$a_t \;=\; \alpha \times L \;=\; 12.7\ {\rm rad/s}^2 \times 1.00\ {\rm m} \;=\; 12.7\ {\rm m/s}^2. \tag{48}$$

Note that this acceleration is bigger than the free fall acceleration g!

(e) Algebraically, the rod's weight mg has lever arm

$$\ell \;=\; \tfrac{1}{2}L \times \cos\theta \tag{49}$$

and hence torque

$$\tau(mg) \;=\; mg \times \tfrac{1}{2}L\cos\theta \;=\; \frac{\cos\theta}{2} \times mgL. \tag{50}$$

This is also the net torque on the rod,

$$\tau^{\rm net} \;=\; \tau(mg) \;=\; \frac{\cos\theta}{2} \times mgL. \tag{51}$$

The rod's moment of inertia is

$$I = \frac{mL^2}{3}, \tag{52}$$

so the rod's angular acceleration is

$$\alpha = \frac{\tau^{\text{net}}}{I} = \frac{\cos\theta}{2} \times mgL \Bigg/ \frac{mL^2}{3} = \frac{3\cos\theta}{2} \times \frac{g}{L}. \tag{53}$$

Note that the rod's mass m cancels out from this formula.

When the rod just begins to move, its free end has only the tangential acceleration

$$a = a_t = \alpha \times L = \frac{3\cos\theta}{2} \times \frac{g}{L} \times L = \frac{3\cos\theta}{2} \times g. \tag{54}$$

At this point, the rod's length L also cancels out, so the free end's acceleration depends only on g and θ. For $\cos\theta > \frac{2}{3}$ — *i.e.*, for $\theta < \arccos\frac{2}{3} = 48^\circ$ — this acceleration is higher than the free fall acceleration g!