Non-textbook problem #**I**:

In the Crater Lake, the water is uniformly cold (about 4° C), so the gas inside the bubble stays at constant temperature as the bubble floats from the bottom to the surface. Hence, the volume of the bubble is related to the absolute pressure by the Boyle's law (sometimes called the Boyle–Mariotte law):

$$P \times V = \text{const}, \quad \text{provided temperature} = \text{const}. \tag{1}$$

As the bubble floats from the bottom of the lake to its surface, the gas pressure inside the bubble changes with the water pressure outside it,[⋆] so the volume of the bubble changes to keep the $P \times V$ product constant. Comparing the volume V_b of the bubble that has formed near the bottom to the volume V_s of the same bubble when it has floated to the surface, we have

$$P_b \times V_b = P_s \times V_s \quad \Longrightarrow \quad \frac{V_s}{V_b} = \frac{P_b}{P_s} \tag{2}$$

where P_b is the absolute water pressure at the bottom and P_s the absolute water pressure at the surface.

At the surface, the absolute water pressure follows from the air pressure at the lake's altitude,

$$P_s = P_{\text{atm}} = 81 \text{ kPa}. \tag{3}$$

At the bottom, the pressure is much higher due to the weight of all the water. According to

⋆ Strictly speaking, there is a small difference between the gas pressure inside a bubble and the water pressure outside it due to tension of the bubble's surface. This difference is significant for microscopically small bubbles but becomes small for larger bubbles. For the bubble in question, the surface tension's effect is too small to matter, so I am neglecting it altogether.

the hydrostatic equation, at depth $d = 594$ m,

$$P_b - P_s = \rho_{\text{water}} \times g \times d = 1\underline{000}\ \text{kg/m}^3 \times 9.8\ \text{N/kg} \times 594\ \text{m} = 5\,820\,000\ \text{Pa} = 5820\ \text{kPa}, \tag{4}$$

so the absolute water pressure at the bottom is

$$P_b = P_s + \rho g d = 81\ \text{kPa} + 5820\ \text{kPa} \approx 5900\ \text{kPa}. \tag{5}$$

Plugging these absolute pressures into eq. (2), we obtain

$$\frac{V_s}{V_b} = \frac{P_b}{P_s} = \frac{5900\ \text{kPa}}{81\ \text{kPa}} \approx 73 \quad \Longrightarrow \quad V_s \approx 73 \times V_b = 73\ \text{mm}^3. \tag{6}$$

Non-textbook problem #**II**:

The net vertical force on a balloon going up to the sky must be positive,

$$F_y^{\text{net}} = F_B - W_{\text{total}} > 0. \tag{7}$$

where F_B is the buoyant force on the balloon and W_{total} is the total weight of the balloon, including the hot air inside it. Treating the hot air's weight as a separate item and combining the weight of everything else into the *gross weight*, we have

$$F_B - W_{\text{hot air}} - W_{\text{gross}} > 0, \tag{8}$$

hence

$$W_{\text{gross}} < W_{\text{gross}}^{\text{max}} = F_B - W_{\text{hot air}} \tag{9}$$

— the maximal gross weight a balloon can lift is the difference between the buoyant force and and the weight of hot air inside the balloon.

Given the density ρ_{in} of the hot air inside the balloon, its weight follows from the balloon's volume V:

$$W_{\text{hot air}} = g \times M_{\text{hot air}} = g \times V \times \rho_{\text{in}} . \tag{10}$$

The buoyant force on balloon is the weight of the cold air it displaces. Given the balloon's volume V and the density ρ_{out} of the cold air outside the balloon, the buoyant force follows as

$$F_B = g \times V \times \rho_{\text{out}} . \tag{11}$$

Consequently, eq. (9) gives us the maximal gross weight

$$W_{\text{gross}}^{\text{max}} = F_B - W_{\text{hot air}} = g \times V \times \rho_{\text{out}} - g \times V \times \rho_{\text{in}} = g \times V \times \big(\rho_{\text{out}} - \rho_{\text{in}}\big). \tag{12}$$

Note that it's the difference between air densities outside and inside the balloon that limits its gross weight.

Numerically, the maximal gross weight of the balloon in question is

$$W_{\text{gross}}^{\text{max}} = 9.8\ \text{N/kg} \times 1800\ \text{m}^3 \times \big(1.18\ \text{kg/m}^3 - 0.91\ \text{kg/m}^3 = 0.27\ \text{kg/m}^3\big) \approx 4800\ \text{N} \tag{13}$$

or about 1100 pounds.

Non-textbook problem **#III**:

Weighing the bowl in the air gives its true weight $W^{\text{true}} = Mg$. But when the bowl is weighed under water, its apparent weight is reduced by the buoyant force, thus

$$\begin{aligned} W^{\text{app}} &= W^{\text{true}} - F^{\text{buoyant}} \\ &= Mg - V \times \rho_{\text{water}} \times g \end{aligned} \tag{14}$$

where V is the volume of water displaced by the bowl. When the bowl is completely submerged, the empty space inside the bowl is filled by water and only the glass itself displaces the water. In term of eq. (14), this means that V is the volume occupied by the glass, which is related to the bowl's mass as

$$V = \frac{M}{\rho_{\text{glass}}}. \tag{15}$$

Consequently,

$$W^{\text{app}} = Mg - \frac{M}{\rho_{\text{glass}}} \times \rho_{\text{water}} \times g = Mg \times \left(1 - \frac{\rho_{\text{water}}}{\rho_{\text{glass}}}\right), \tag{16}$$

or in other words

$$\frac{W^{\text{app}}_{\text{in water}}}{W_{\text{in air}}} = 1 - \frac{\rho_{\text{water}}}{\rho_{\text{glass}}}. \tag{17}$$

Inverting this formula gives us

$$\frac{\rho_{\text{water}}}{\rho_{\text{glass}}} = 1 - \frac{W^{\text{app}}_{\text{in water}}}{W_{\text{in air}}}. \tag{18}$$

For the bowl in question

$$\frac{\rho_{\text{water}}}{\rho_{\text{glass}}} = 1 - \frac{3 \text{ lb}}{5 \text{ lb}} = 1 - 0.6 = 0.4, \tag{19}$$

hence,

$$\rho_{\text{glass}} = \frac{\rho_{\text{water}}}{0.4} = \frac{1 \text{ g/cm}^3}{0.4} = 2.5 \text{ g/cm}^3 = 2500 \text{ kg/m}^3. \tag{20}$$

Non-textbook problem #**IV**:

For simplicity, suppose the glass is a perfect cylinder of horizontal section $A = \pi R^2$. Then the water level in the glass is given by

$$H = \frac{1}{A} \times \left(V_{\text{liq}} + V_{\text{disp}}\right) \tag{21}$$

where V_{liq} is the volume of liquid water in the glass, and V_{disp} is the volume of water displaced by the ice cube. By Archimedes's law, the latter volume governs the buoyant force on the cube, and since the cube floats without sinking or rising, this buoyant force must be equal to the cube's weight:

$$V_{\rm disp} \times \rho_{\rm liq.water} \times g \;=\; M_{\rm ice} \times g, \tag{22}$$

hence

$$V_{\rm disp} \;=\; \frac{M_{\rm cube}}{\rho_{\rm liq.water}}\,. \tag{23}$$

Note that this formula involves the density of the liquid water rather than of the ice because we don't want the volume of the whole cube but only of its submerged part that displaces water.

Combining eq. (23) with an obvious formula

$$V_{\rm liq} \;=\; \frac{M_{\rm liq}}{\rho_{\rm liq.water}}\,, \tag{24}$$

we find

$$V_{\rm liq} \;+\; V_{\rm disp} \;=\; \frac{M_{\rm liq} \;+\; M_{\rm ice}}{\rho_{\rm liq.water}} \;=\; \frac{M_{\rm total\,H_2O}}{\rho_{\rm liq.water}} \tag{25}$$

and hence via eq. (21),

$$H \;=\; \frac{M_{\rm total\,H_2O}}{A \times \rho_{\rm liq.water}}\,. \tag{26}$$

In other words, the water level depends only on the total mass of H_2O in both liquid and solid form, and it does not matter how much H_2O is frozen and how much is liquid, as long as all the ice floats on liquid water. Therefore, when the ice cube melts and turns into more liquid water, the water level stays exactly the same because the total mass of H_2O remain unchanged.

Now suppose the glass is not cylindrical but has a different shape. In this case, the water level is a more complicated function of $V_{\rm liq} + V_{\rm disp}$, but it's still a function of the same sum of liquid volume and displaced volume. According to eq. (25), this sum remains unchanged when the ice cube melts, hence regardless of the glass's shape, the water level remains unchanged.

Non-textbook problem #**V**:

A steady flow of a fluid in a pipe is governed by two conservation rules. First, the flow rate is constant — what flows into the pipe must flow out of it,

$$\mathcal{F} \;=\; v \times A \;=\; \text{const} \tag{27}$$

where A is the cross-section area of the pipe and v is the speed of the flow (averaged over the pipe's cross-section). The second rule is the Bernoulli equation

$$P \;+\; \rho g y \;+\; \tfrac{1}{2}\rho v^2 \;=\; \text{const}, \tag{28}$$

which follows from the work-energy theorem for the fluid.

(a) The flow rate through the narrow and the wide parts of the pipe must be the same, thus

$$A_n \times v_n \;=\; \mathcal{F} \;=\; A_w \times v_w\,. \tag{29}$$

Solving this equation for the water speed v_n in the narrow part gives us

$$v_n \;=\; v_w \times \frac{A_w}{A_n}\,, \tag{30}$$

where the ratio of cross-sectional areas follows from the ratio of diameters:

$$\frac{A_w}{A_n} \;=\; \frac{\pi R_w^2}{\pi R_n^2} \;=\; \left(\frac{R_w}{R_n}\right)^2 \;=\; \left(\frac{2R_w = 2\text{ cm}}{2R_n = 1\text{ cm}}\right)^2 \;=\; 2^2 \;=\; 4, \tag{31}$$

hence

$$v_n \;=\; v_w \times 4 \;=\; 11\text{ m/s} \times 4 \;=\; 44\text{ m/s}. \tag{32}$$

(b) Now let's apply the Bernoulli equation (28) to the wide and narrow parts of the pipe:

$$P_w \;+\; \rho g y_w \;+\; \tfrac{1}{2}\rho v_w^2 \;=\; P_n \;+\; \rho g y_n \;+\; \tfrac{1}{2}\rho v_n^2\,. \tag{33}$$

The pipe in question is horizontal, so $y_w = y_n$ and the $\rho g y$ terms on the two sides of eq. (33) cancel each other. This leaves us with

$$P_w \;+\; \tfrac{1}{2}\rho v_w^2 \;=\; P_n \;+\; \tfrac{1}{2}\rho v_n^2 \tag{34}$$

and hence

$$\begin{aligned} P_w \;-\; P_n \;&=\; \tfrac{1}{2}\rho v_n^2 \;-\; \tfrac{1}{2}\rho v_w^2 \;=\; \frac{\rho}{2}\times\bigl(v_n^2 \;-\; v_w^2\bigr) \\ &=\; \frac{1000\ \mathrm{kg/m^3}}{2}\times\Bigl((44\ \mathrm{m/s})^2 \;-\; (11\ \mathrm{m/s})^2\Bigr) \\ &=\; 908\ \mathrm{kPa}. \end{aligned} \tag{35}$$

The narrow part of the pipe opens to the atmosphere, so the pressure there is the atmospheric pressure,

$$P_n \;=\; P_{\mathrm{atm}} \;=\; 101\ \mathrm{kPa}. \tag{36}$$

In the wide part of the pipe, the pressure is higher by the amount calculated in eq. (35), so its value must be

$$P_w \;=\; P_n + \bigl(P_w - P_n\bigr) \;=\; 101\,\mathrm{kPa} + 908\,\mathrm{kPa} \;=\; 1009\,\mathrm{kPa} \;\approx\; 1010\,\mathrm{kPa} \;\approx\; 10\ \mathrm{atm} \tag{37}$$

Note: this is the *absolute pressure* in the wide part of the pipe. The gauge pressure there is lower by 100 kPa,

$$p_w^{\mathrm{gauge}} \;=\; P_w \;-\; P_{\mathrm{atm}} \;=\; 908\ \mathrm{kPa}. \tag{38}$$

Non-textbook problem #**VI**:

Far ahead of the plane, the undisturbed air is at rest or moves with some wind velocity $\vec{v}_w$ relative to the ground. The plane moves with some velocity $\vec{v}_p$ relative to this wind, which means that *relative to the plane, the air far ahead of the plane moves towards it with velocity* $-\vec{v}_p$.

Let's work in the reference frame of the plane and consider how the air flows around it. Far ahead of the plane, there is a single stream of air flowing at speed v_p, but closer to the plane this stream splits into two — one stream goes above the plane's wings and the other stream below the wings. Each stream has a fixed geometry and a steady flow, so each stream satisfies its own Bernoulli equation

$$P\;+\;\rho g y\;+\;\tfrac{1}{2}\rho v^2\;=\;\text{const}\quad\text{(along a stream).}\tag{28}$$

In particular, comparing the flow far ahead of the plane with the flow immediately above or below the plane's wings, we have

$$\begin{aligned} P^{\text{wing top}}\;+\;\rho g y^{\text{wing top}}\;+\;\tfrac{1}{2}\rho(v^{\text{wing top}})^2\;&=\;P^{\text{ahead,top}}\;+\;\rho g y^{\text{ahead,top}}\;+\;\tfrac{1}{2}\rho(v^{\text{ahead,top}})^2,\\ P^{\text{wing bot}}\;+\;\rho g y^{\text{wing bot}}\;+\;\tfrac{1}{2}\rho(v^{\text{wing bot}})^2\;&=\;P^{\text{ahead,bot}}\;+\;\rho g y^{\text{ahead,bot}}\;+\;\tfrac{1}{2}\rho(v^{\text{ahead,bot}})^2.\end{aligned}\tag{39}$$

To simplify these equations, we assume that the plane flies horizontally and neglect the small vertical motion of the air within a stream. In other words, we take

$$y^{\text{ahead,top}}\;\approx\;y^{\text{wing top}}\quad\text{and}\quad y^{\text{ahead,bot}}\;\approx\;y^{\text{wing bot}},\tag{40}$$

so that the $\rho g y$ terms in eqs. (39) cancel out between two sides of each equation. This gives us somewhat simpler equations

$$\begin{aligned} P^{\text{wing top}}\;+\;\tfrac{1}{2}\rho(v^{\text{wing top}})^2\;&=\;P^{\text{ahead,top}}\;+\;\tfrac{1}{2}\rho(v^{\text{ahead,top}})^2,\\ P^{\text{wing bot}}\;+\;\tfrac{1}{2}\rho(v^{\text{wing bot}})^2\;&=\;P^{\text{ahead,bot}}\;+\;\tfrac{1}{2}\rho(v^{\text{ahead,bot}})^2.\end{aligned}\tag{41}$$

for the two streams.

(a) The air flow above the plane's wings is governed by the first eq. (41). Far ahead of the plane, the pressure is simply the atmospheric pressure at the plane's altitude

$$P^{\text{ahead,top}} = P^{\text{atm}} = 78\ \text{kPa}, \tag{42}$$

while the speed of the air relative to the plane is the plane's airspeed

$$v^{\text{ahead,top}} = v_p = 63\ \text{m/s}. \tag{43}$$

On top of each wing, the air flows at faster speed

$$v^{\text{wing top}} = 67\ \text{m/s}, \tag{44}$$

so the pressure drops below the atmospheric. Specifically, according to the first eq. (41),

$$\begin{aligned} P^{\text{wing top}} &= P^{\text{ahead,top}} + \tfrac{1}{2}\rho(v^{\text{ahead,top}})^2 - \tfrac{1}{2}\rho(v^{\text{wing top}})^2 \\ &= P^{\text{atm}} + \frac{\rho}{2} \times \left(v_p^2 - (v^{\text{wing top}})^2\right) \\ &= 78\ \text{kPa} + \frac{1.00\ \text{kg/m}^2}{2} \times \left((63\ \text{m/s})^2 - (67\ \text{m/s})^2\right) \\ &= 78\ \text{kPa} - 260\ \text{Pa} \\ &= 77\,740\ \text{Pa}. \end{aligned} \tag{45}$$

Note: only the first 2 figures of the last number are significant, but I keep 2 extra digits to avoid round-off errors in part (c). This is OK because the 78 kPa term will cancel out exactly, so its limited accuracy does not matter.

(b) The air flow below the plane's wings is governed by the second eq. (41). Similar to the other stream (above the wings), the pressure in the stream below the wings is the atmospheric pressure at the plane's altitude,

$$P^{\text{ahead,top}} = P^{\text{atm}} = 78\ \text{kPa}, \tag{46}$$

while the speed of the air relative to the plane is the plane's airspeed

$$v^{\text{ahead,top}} = v_p = 63 \text{ m/s}. \tag{47}$$

However, below the wings the air flows at a slower speed

$$v^{\text{wing top}} = 55 \text{ m/s}, \tag{48}$$

so the pressure rises above the atmospheric. Specifically, according to the second eq. (41),

$$\begin{aligned} P^{\text{wing bot}} &= P^{\text{ahead,bot}} + \tfrac{1}{2}\rho(v^{\text{ahead,bot}})^2 - \tfrac{1}{2}\rho(v^{\text{wing bot}})^2 \\ &= P^{\text{atm}} + \frac{\rho}{2} \times \Big(v_p^2 - (v^{\text{wing bot}})^2\Big) \\ &= 78 \text{ kPa} + \frac{1.00 \text{ kg/m}^2}{2} \times \Big((63 \text{ m/s})^2 - (55 \text{ m/s})^2\Big) \\ &= 78 \text{ kPa} + 472 \text{ Pa} \\ &\approx 78\,470 \text{ Pa}. \end{aligned} \tag{49}$$

Again, only the first 2 figure in this number are significant, but I keep 2 extra digits to avoid round-off errors in part (c).

(c) The air pressure on the bottom sides of the wings is larger than the pressure on the top sides. Both pressures act on the same area A — the net area of the two wings — so the upward force on the wing bottom sides is larger than the downward force on the top sides. This difference creates a net upward force called the *lift force,*

$$F^{\text{lift}} = F^{\text{up}} - F^{\text{down}} = A \times P^{\text{wing bot}} - A \times P^{\text{wing top}} = A \times \Big(P^{\text{wing bot}} - P^{\text{wing top}}\Big). \tag{50}$$

The pressures above and below the wings were obtained in parts (a) and (b). Plugging them into the lift-force equation (50), we obtain

$$F^{\text{lift}} = 16 \text{ m}^2 \times \Big(78\,470 \text{ Pa} - 77\,740 \text{ Pa} = 730 \text{ Pa}\Big) \approx 11\,700 \text{ N} \approx 2600 \text{ lb}. \tag{51}$$

PS: To see how the atmospheric pressure $P^{\rm atm}=78$ kPa cancels out from the lift force on the plane, let's redo the whole calculation algebraically, starting with eqs. (41). Note that far ahead of the plane, both air streams have exactly the same speed

$$v^{\rm ahead,top}\ =\ v^{\rm ahead,bot}\ =\ v_p \tag{52}$$

and the same pressure

$$P^{\rm ahead,top}\ =\ P^{\rm ahead,bot}\ =\ P^{\rm atm} \tag{53*}$$

Thus, on the right hand sides of both equations (41) we have exactly the same combination

$$P^{\rm ahead,top}\ +\ \tfrac12\rho(v^{\rm ahead,top})^2\ =\ P^{\rm ahead,bot}\ +\ \tfrac12\rho(v^{\rm ahead,bot})^2\ =\ P^{\rm atm}\ +\ \tfrac12\rho v_p^2\,, \tag{55}$$

so the left hand sides of the two equations must be equal to each other,

$$P^{\rm wing\,top}\ +\ \tfrac12\rho(v^{\rm wing\,top})^2\ =\ P^{\rm wing\,bot}\ +\ \tfrac12\rho(v^{\rm wing\,bot})^2. \tag{56}$$

From this equation, we immediately obtain the difference between air pressures below and above the wings in terms of the speeds of air flow at those places:

$$P^{\rm wing\,bot}\ -\ P^{\rm wing\,top}\ =\ \frac{\rho}{2}\times\Bigl((v^{\rm wing\,top})^2\ -\ (v^{\rm wing\,bot})^2\Bigr). \tag{57}$$

Consequently, eq. (50) gives us the lift force on the plane:

$$F^{\rm lift}\ =\ A\times\frac{\rho}{2}\times\Bigl((v^{\rm wing\,top})^2\ -\ (v^{\rm wing\,bot})^2\Bigr). \tag{58}$$

Evaluating this formula for the plane in question, we have

$$F^{\rm lift}\ =\ 16\ {\rm m}^2\times\frac{1.00\ {\rm kg/m}^3}{2}\times\bigl((67\ {\rm m/s})^2\ -\ (55\ {\rm m/s})^2\bigr)^2\ =\ 11\,700\ {\rm N}\ \approx\ 2600\ {\rm lb}. \tag{59}$$

In a level flight, this lift force balances the plane's weight Mg, so the plane in question must weigh about 2600 pounds.

⋆ To be precise, there is a small hydrostatic pressure difference between the two air-streams ahead of the plane,

$$P^{\rm ahead,bot}\ -\ P^{\rm ahead,top}\ =\ \rho g\times\bigl(y^{\rm ahead,top}\ -\ y^{\rm ahead,bot}\bigr). \tag{54}$$

But this difference is so small compared to the other pressure differences in this problem that we may safely ignore it.

<u>Non-textbook problem **#VII**:</u>

The soccer ball on the video veers to the side because of the *Magnus effect* which creates a sideways force similar to the lift force on a wing of an airplane, *cf.* the previous problem.

Let's go in the reference frame of the flying ball and consider how the air flows around the ball. Far ahead of the ball, the air flows back — towards the ball — with speed equal to v^{ball}. Closer to the ball, the air speeds up or slows down due to friction against the spinning surface of the ball. Since the air flows backwards (with respect to the ball's direction of flight), it speeds up over the side of the ball which spins back and slows down over the side of the ball which spins forward. This speed-up / slow-down of the air stream lowers / raises the local air pressure according to the Bernoulli's equation, exactly as for the airplane's wings in the previous problem. And this difference in pressure creates a net *Magnus force* on the ball, similar to the lift force on the plane.

The side of the ball which spins forward slows down the air flow, so it acts as a bottom of a wing. The opposite side — which spins back — speeds up the air flow, so it acts as a top of the wing. So by analogy with the lift force, the Magnus force pushes the ball in the direction of the side which spins back.

In baseball, a batter wishing for a home run hits the ball below the center. This gives the ball a back spin about a horizontal axis — the top of the ball spins back while the bottom spins forward. Consequently, the Magnus force is directed *up* and creates a lift, which drastically increases the horizontal range of the ball.

In soccer, to make the ball veer left or right, one needs a horizontal Magnus force. This calls for a ball spinning around a vertical axis. If the spin is clockwise (as viewed from above), the left side of the ball spins forward while the right side spins back; this creates for a Magnus force directed to the right (relative to the ball's direction), and the ball veers right, as in the video. If the ball spins counterclockwise, it veers left.

To make the ball veer right, the player should kick it left from center. The force of such a kick creates a clockwise torque *relative to the ball's center of mass*, which makes the ball spin clockwise as it flies forward. Consequently, the Magnus force will make the ball veer to the right.