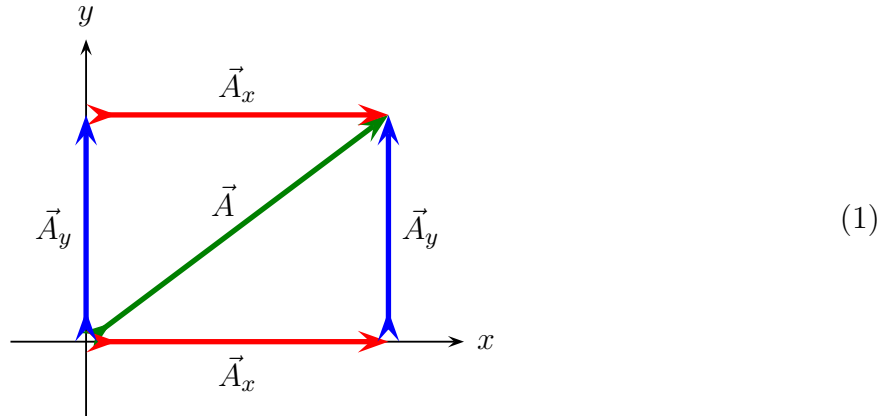


COMPONENTS OF VECTORS

To describe motion in two dimensions we need a coordinate system with two perpendicular axes, x and y . In such a coordinate system, any vector \vec{A} can be uniquely decomposed into a sum of two perpendicular vectors $\vec{A} = \vec{A}_x + \vec{A}_y$ where \vec{A}_x is parallel to the x axis while \vec{A}_y is parallel to the y axis, for example



Since the \vec{A}_x vector is always parallel to the x axis, we may describe it by a single signed number A_x , which is positive when \vec{A}_x points right but negative when \vec{A}_y points left. Likewise, the \vec{A}_y vectors may be described by a single signed number A_y — positive when \vec{A}_y points up but negative when \vec{A}_y points down. The two signed numbers A_x and A_y are called the *components* of the vector \vec{A} . In two dimensions, any vector \vec{V} can be completely specified by its components (V_x, V_y) .

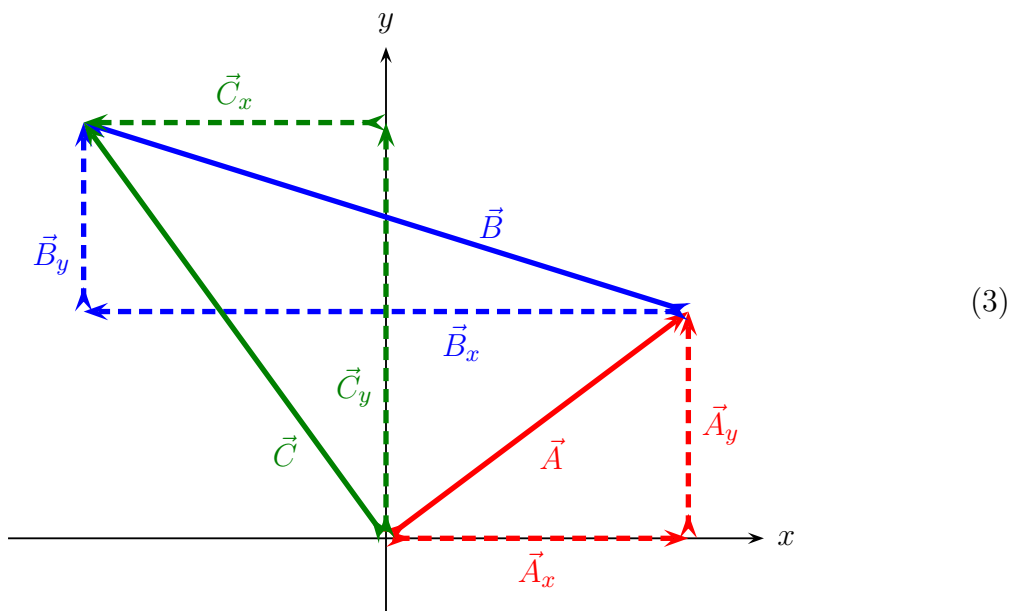
Describing motion in all 3 dimensions of space requires a coordinate system with 3 perpendicular axes (x, y, z) . Consequently, a 3D vector \vec{V} has three components (V_x, V_y, V_z) , and we need to know all 3 components to completely specify the vector.

ADDING VECTORS IN COMPONENTS.

In component notations, adding vectors is very easy: The components of a vector sum $\vec{C} = \vec{A} + \vec{B}$ are simply the algebraic sums

$$\begin{aligned} C_x &= A_x + B_x, \\ C_y &= A_y + B_y. \end{aligned} \tag{2}$$

Here is the geometric explanation of this rule:



In the same way, we may sum up several vectors: To get the components of a vector sum

$$\vec{C} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \cdots + \vec{A}_n, \tag{4}$$

we separately sum up the x components of all the vectors and the y components of all the vectors:

$$\begin{aligned} C_x &= A_{1x} + A_{2x} + A_{3x} + \cdots + A_{nx}, \\ C_y &= A_{1y} + A_{2y} + A_{3y} + \cdots + A_{ny}. \end{aligned} \tag{5}$$

Note: the sums here are algebraic, so please mind the \pm signs of the components.

For the 3D vectors, there are similar formulae, but there is one more algebraic sum for the z components:

$$\begin{aligned} C_x &= A_{1x} + A_{2x} + A_{3x} + \cdots + A_{nx}, \\ C_y &= A_{1y} + A_{2y} + A_{3y} + \cdots + A_{ny}, \\ C_z &= A_{1z} + A_{2z} + A_{3z} + \cdots + A_{nz}. \end{aligned} \tag{6}$$

Vector subtraction in components works similar to vector addition. To get

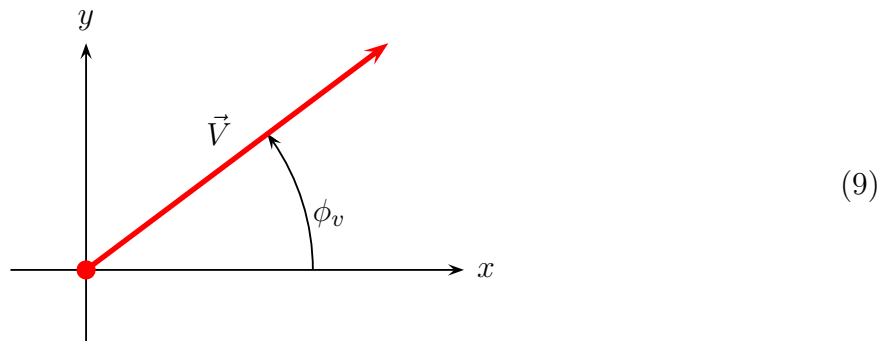
$$\vec{B} = \vec{C} - \vec{A} \tag{7}$$

in components, subtract the components of \vec{A} from the components of \vec{C} ,

$$\begin{aligned} B_x &= C_x - A_x, \\ B_y &= C_y - A_y, \\ \text{in 3D also } B_z &= C_z - A_z. \end{aligned} \tag{8}$$

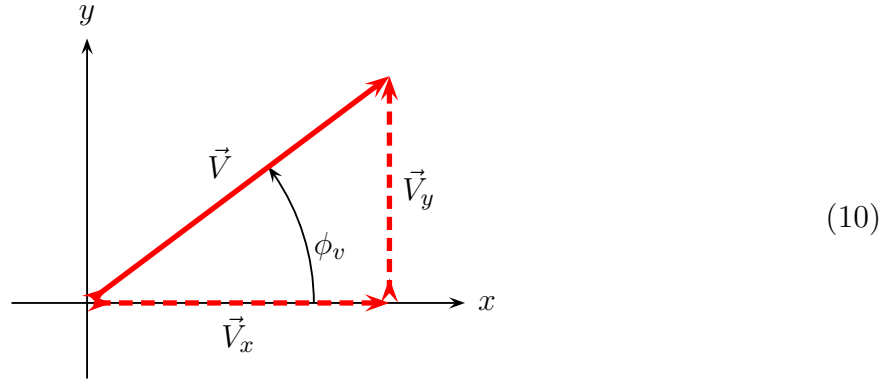
CONVERSION FROM MAGNITUDE AND DIRECTION TO COMPONENTS.

A vector quantity \vec{V} has magnitude and direction. On a graph the magnitude is shown by the length of the arrowed line; algebraically, the magnitude is a non-negative number $|\vec{V}|$. In a 2D plane, the direction of a vector can be specified by the angle ϕ_v it makes with the x axis, for example



Now let us draw a similar diagram which also includes the components (V_x, V_y) of the

vector \vec{V} :



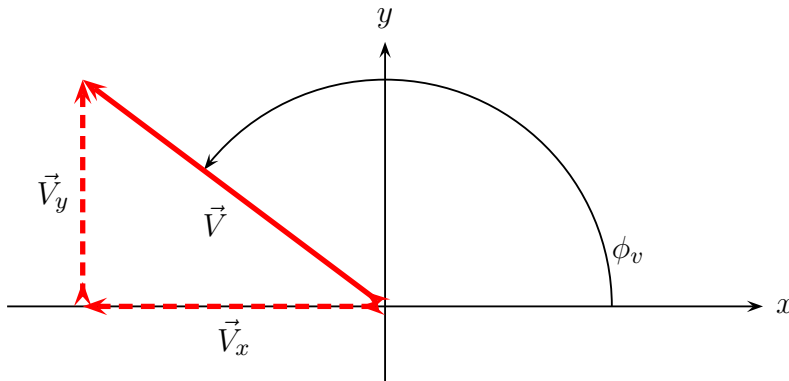
Note the right triangle made by the three red lines; taking the ratios of this triangle's sides and applying basic trigonometry, we immediately obtain

$$\frac{V_x}{|\vec{V}|} = \cos \phi_v, \quad \frac{V_y}{|\vec{V}|} = \sin \phi_v, \quad (11)$$

and therefore

$$V_x = |\vec{V}| \times \cos \phi_v, \quad V_y = |\vec{V}| \times \sin \phi_v. \quad (12)$$

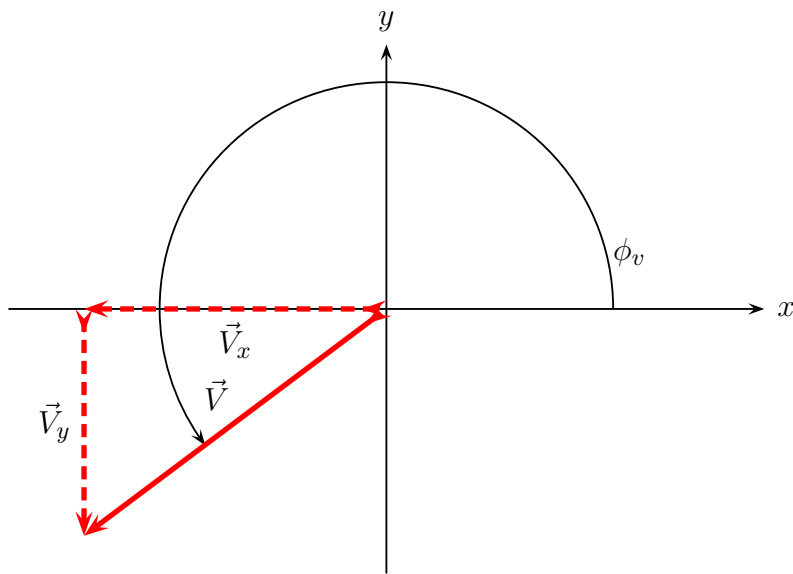
The triangle on the diagram (10) is drawn for direction of \vec{V} in the first quadrant of the coordinate system (between the positive x and positive y direction, $\phi_v < 90^\circ$), but the formulae (12) for the components work for all possible directions, provided we always measure the angle ϕ_v counterclockwise from the positive x axis. For example, for \vec{V} in the second quadrant



$$\begin{aligned} 90^\circ < \phi_v < 180^\circ, \\ V_x &= |\vec{V}| \times \cos \phi_v < 0, \\ V_y &= |\vec{V}| \times \sin \phi_v > 0, \end{aligned}$$

(13)

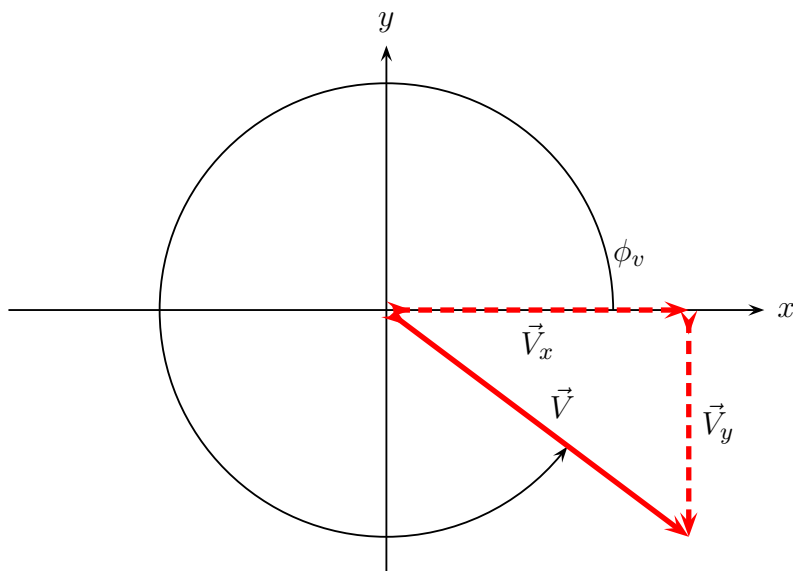
Likewise, for \vec{V} in the third quadrant



$$\begin{aligned} 180^\circ < \phi_v < 270^\circ, \\ V_x &= |\vec{V}| \times \cos \phi_v < 0, \\ V_y &= |\vec{V}| \times \sin \phi_v < 0, \end{aligned}$$

(14)

or for \vec{V} in the fourth quadrant



$$\begin{aligned} 270^\circ < \phi_v < 360^\circ, \\ V_x &= |\vec{V}| \times \cos \phi_v > 0, \\ V_y &= |\vec{V}| \times \sin \phi_v < 0, \end{aligned}$$

(15)

CONVERSION FROM COMPONENTS TO MAGNITUDE AND DIRECTION.

Now supposed we know the (V_x, V_y) components of a vector \vec{V} ; how do we find the vector's magnitude and direction?

The magnitude $|\vec{V}|$ follows from the Pythagoras theorem for the right triangles on any of the diagrams on the last two pages:

$$|\vec{V}|^2 = V_x^2 + V_y^2 \quad (16)$$

— regardless of the signs of the V_x and V_y — and therefore

$$|\vec{V}| = \sqrt{V_x^2 + V_y^2}. \quad (17)$$

To find the direction of \vec{V} , we need a bit of trigonometry. Let's take the ratio of the two equations (12) for the components (V_x, V_y) :

$$\frac{V_y}{V_x} = \frac{|\vec{V}| \times \sin \phi_v}{|\vec{V}| \times \cos \phi_v} = \frac{\sin \phi_v}{\cos \phi_v} = \tan \phi_v. \quad (18)$$

Thus the ratio V_y/V_x gives us the tangent of the angle ϕ_v , so *naively* we may calculate the angle ϕ_v itself as the arc-tangent (the inverse tangent) of this ratio,

$$\phi_v \stackrel{??}{=} \arctan \frac{V_y}{V_x}. \quad (19)$$

However, the formula may be off by 180° , so it might give us precisely the opposite direction. Indeed, the vectors \vec{V} and $-\vec{V}$ have opposite directions but similar ratios

$$\frac{V_y}{V_x} = \frac{-V_y}{-V_x}. \quad (20)$$

This ambiguity is related to the trigonometric identity

$$\text{for any angle } \varphi, \quad \tan(\varphi) = \tan(\varphi \pm 180^\circ). \quad (21)$$

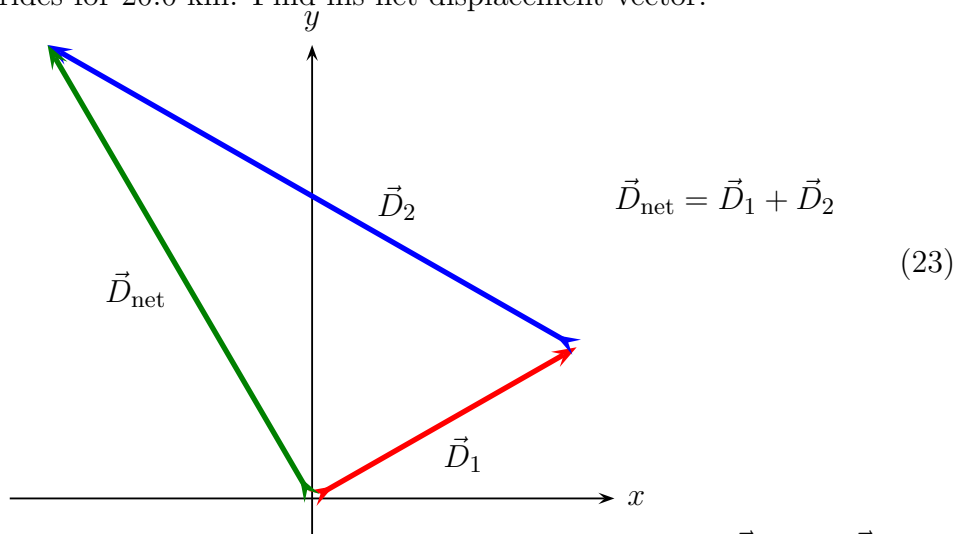
Therefore, given the components of a vector, its direction is

$$\phi_v = \mathbf{either} \arctan \frac{V_y}{V_x} \quad \mathbf{or} \quad \arctan \frac{V_y}{V_x} \pm 180^\circ, \quad (22)$$

depending on the signs of the V_x and V_y .

USING CONVERSION TO ADD VECTORS.

Consider a simple problem: A person rides a bike for 10.0 km in the direction 30° (counterclockwise from the x axis), then changes direction to 150° (also counterclockwise from the x axis) and rides for 20.0 km. Find his net displacement vector.



To solve this problem, we start by converting the displacement vectors \vec{D}_1 and \vec{D}_2 into components:

$$\vec{D}_1 = (10.0 \text{ km}; 30^\circ) \implies \begin{cases} D_{1x} = 10.0 \text{ km} \times \cos(30^\circ) = +8.66 \text{ km}, \\ D_{1y} = 10.0 \text{ km} \times \sin(30^\circ) = +5.00 \text{ km}, \end{cases} \quad (24)$$

$$\vec{D}_2 = (20.0 \text{ km}; 150^\circ) \implies \begin{cases} D_{2x} = 20.0 \text{ km} \times \cos(150^\circ) = -17.32 \text{ km}, \\ D_{2y} = 20.0 \text{ km} \times \sin(150^\circ) = +10.00 \text{ km}. \end{cases}$$

Next, we add the two vectors in components:

$$\begin{aligned} D_x^{\text{net}} &= D_{1x} + D_{2x} = +8.66 \text{ km} - 17.32 \text{ km} = -8.66 \text{ km}, \\ D_y^{\text{net}} &= D_{1y} + D_{2y} = +5.00 \text{ km} + 10.00 \text{ km} = +15.00 \text{ km}. \end{aligned} \quad (25)$$

Finally, we convert the components of the net displacement vector into its direction and

magnitude. For the magnitude we have

$$\begin{aligned} |\vec{D}_{\text{net}}|^2 &= (D_x^{\text{net}})^2 + (D_y^{\text{net}})^2 = (-8.66 \text{ km})^2 + (+15.00 \text{ km})^2 \\ &= 75 \text{ km}^2 + 225 \text{ km}^2 = \underline{300 \text{ km}^2} \end{aligned} \quad (26)$$

and hence

$$|\vec{D}_{\text{net}}| = \sqrt{300 \text{ km}^2} \approx 17.3 \text{ km}. \quad (27)$$

As to the direction,

$$\frac{D_y^{\text{net}}}{D_x^{\text{net}}} = \frac{+15.00 \text{ km}}{-8.66 \text{ km}} = -1.73 \implies \arctan \frac{D_y^{\text{net}}}{D_x^{\text{net}}} = -60^\circ. \quad (28)$$

The -60° angle is the same as $360^\circ - 60^\circ = +300^\circ$, which correspond to the direction of \vec{D}_{net} being in the fourth quadrant. However, the signs of the components $D_x^{\text{net}} < 0$, $D_y^{\text{net}} > 0$ show that the direction of \vec{D}_{net} is in the second quadrant. This means that the arc-tangent is off by 180° , so the correct direction of the net displacement vector is

$$\phi_{\text{net}} = -60^\circ + 180^\circ = +120^\circ \quad (\text{counterclockwise from the } x \text{ axis}). \quad (29)$$

The above example illustrate a general rule for calculating sums of several vectors,

$$\vec{A}_{\text{net}} = \vec{A}_1 + \vec{A}_2 + \cdots + \vec{A}_n. \quad (30)$$

(1) First, convert all the vectors into components,

$$A_{i,x} = |\vec{A}_i| \times \cos \phi_i, \quad A_{i,y} = |\vec{A}_i| \times \sin \phi_i, \quad \text{for } i = 1, 2, \dots, n. \quad (31)$$

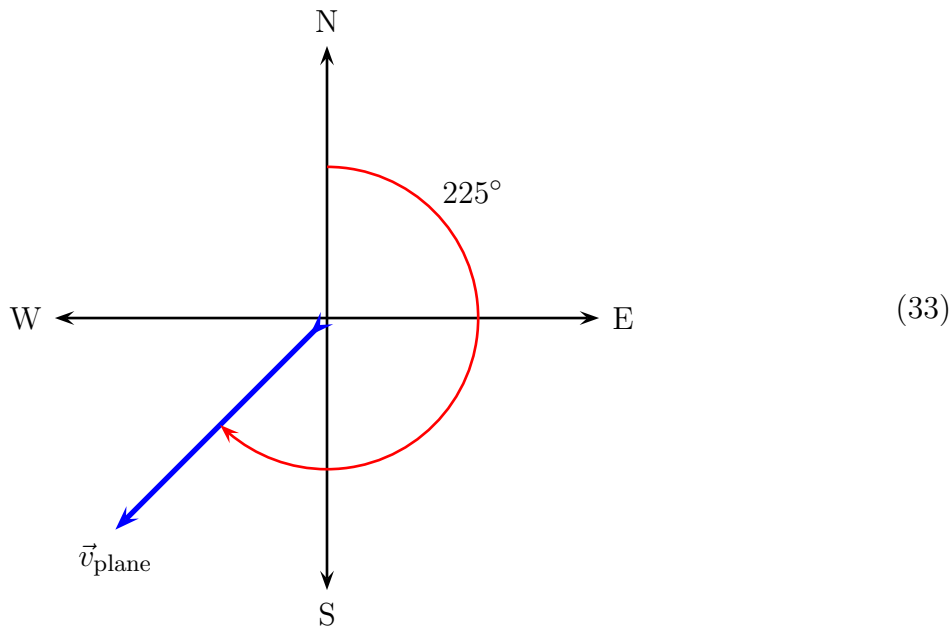
(2) Second, add the vectors in components,

$$A_x^{\text{net}} = A_{1,x} + A_{2,x} + \cdots + A_{n,x}, \quad A_y^{\text{net}} = A_{1,y} + A_{2,y} + \cdots + A_{n,y}. \quad (32)$$

(3) Finally, convert the components of the \vec{A}_{net} into its magnitude and direction.

NAVIGATION CONVENTION

In navigation, the directions are given by angles counted clockwise from North instead of counterclockwise from the x axis (whatever that might be). For example, an airplane flying in the Southwest direction — which is 225° clockwise from North — is said to have heading 225° .



To avoid confusion when you work a navigation-related problem, it is best to avoid the x and y axes altogether and use the North and East axes instead of them. In particular, for the vectors you should use the North and East components instead of the x and y components. In this convention,

$$V_N = |\vec{V}| \times \cos \phi_v, \quad V_E = |\vec{V}| \times \sin \phi_v, \quad \text{for } \phi_v \text{ counted clockwise from North.} \quad (34)$$

For example, for a plane flying at speed $v = 220$ MPH in the Southwest direction ($\phi_v = 225^\circ$), the velocity vector \vec{v} has components

$$\begin{aligned} v_N &= 220 \text{ MPH} \times \cos 225^\circ = -155 \text{ MPH}, \\ v_E &= 220 \text{ MPH} \times \sin 225^\circ = -155 \text{ MPH}. \end{aligned} \quad (35)$$

Note negative signs of the components since the plane is flying South and West instead of North and East.

MULTIPLYING VECTORS BY SCALARS.

The product of a scalar s and a vector \vec{V} is a vector $s\vec{V}$. Its magnitude is the absolute value of s times the magnitude of \vec{V} ,

$$|s\vec{V}| = |s| \times |\vec{V}|. \tag{36}$$

The direction of the product $s\vec{V}$ is the same as direction of \vec{V} for positive s but opposite from the direction of \vec{V} for negative s . (For $s = 0$ the product $s\vec{V} = \vec{0}$ — the zero vector — and its direction is undefined.)

In components, $\vec{B} = s\vec{A}$ has

$$B_x = s \times A_x, \quad B_y = s \times A_y, \quad \text{and in 3D also } B_z = s \times A_z. \tag{37}$$

The product of a vector and a scalar obeys the usual algebraic rules for opening parentheses:

$$\begin{aligned} s(\vec{V}_1 + \vec{V}_2) &= s\vec{V}_1 + s\vec{V}_2, \\ s(\vec{V}_1 - \vec{V}_2) &= s\vec{V}_1 - s\vec{V}_2, \\ (s_1 + s_2)\vec{V} &= s_1\vec{V} + s_2\vec{V}, \\ (s_1 - s_2)\vec{V} &= s_1\vec{V} - s_2\vec{V}, \\ s_1(s_2\vec{V}) &= (s_1s_2)\vec{V}, \\ \dots\dots\dots \end{aligned} \tag{38}$$

PHYSICAL EXAMPLE: For a motion at constant velocity vector \vec{v} — *i.e.*, motion at constant speed in a constant direction — the displacement vector after time t is given by

$$\vec{D} = t\vec{v}. \tag{39}$$

In terms of the time-dependent position vector (AKA radius-vector) $\vec{R}(t)$ of the moving body — whose components $R_x(t)$ and $R_y(t)$ are simply the time-dependent coordinates (x, y) of

the body — the displacement vector is the vector difference

$$\vec{D} = \Delta\vec{R} = \vec{R}(t) - \vec{R}_0, \quad (40)$$

so eq. (39) becomes

$$\vec{R}(t) = \vec{R}_0 + t\vec{v}. \quad (41)$$

In components, this formula means

$$x(t) = x_0 + t \times v_x \quad \text{and} \quad y(t) = y_0 + t \times v_y \quad (42)$$

— uniform motion in both x and y directions.

DIVISION: You cannot divide a scalar by vector, or a vector by another vector. However, you can divide a vector by a scalar — simply multiply the vector by the inverse scalar,

$$\frac{\vec{V}}{s} \stackrel{\text{def}}{=} \frac{1}{s}\vec{V}. \quad (43)$$

For example, to find the average velocity vector of some body, divide its displacement vector by the time this displacement took,

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{R}}{\Delta t}. \quad (44)$$

In the limit of a very short time interval, this formula gives you the *instantaneous velocity vector*

$$\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{R}(t + \Delta t) - \vec{R}(t)}{\Delta t}. \quad (45)$$

In general, the velocity vector changes with time, which leads to the *acceleration vector*

$$\vec{a}(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t}. \quad (46)$$

MOTION AT CONSTANT ACCELERATION.

Consider motion of some body having a constant acceleration vector \vec{a} . (Neither direction nor magnitude of \vec{a} changes with time.) The velocity vector of such a body changes with time according to

$$\vec{v}(t) = \vec{v}_0 + t\vec{a} \quad (47)$$

where \vec{v}_0 is the initial velocity vector at time $t_0 = 0$. Note: eq. (46) looks simple in vector notations, or in components

$$v_x(t) = v_{0x} + t \times a_x, \quad v_y(t) = v_{0y} + t \times a_y, \quad (48)$$

but it leads to rather complicated formulae for the time dependence of the speed $|\vec{v}(t)|$ and the direction of motion.

The time-dependent position vector $\vec{R}(t)$ of a uniformly accelerating body is given by

$$\vec{R}(t) = \vec{R}_0 + t\vec{v}_0 + \frac{1}{2}t^2\vec{a}, \quad (49)$$

or in components

$$\begin{aligned} x(t) &= x_0 + t \times v_{0x} + \frac{1}{2}t^2 \times a_x, \\ y(t) &= y_0 + t \times v_{0y} + \frac{1}{2}t^2 \times a_y. \end{aligned} \quad (50)$$

PROJECTILE MOTION

A projectile is an object you shoot, kick, throw, or otherwise send flying towards a target, for example a basketball, a bullet, or a grenade. In physics, projectile motion is a motion of a body that has been released with some initial velocity (which generally has both horizontal and vertical components) and then flies free from forces other than gravity and air resistance. When the air resistance may be neglected — which is the only case we shall study in this class — the projectile has a constant acceleration vector $\vec{a} = \vec{g}$ due to gravity. Consequently, the projectile's velocity vector $\vec{v}(t)$ and position vector $\vec{R}(t)$ evolve with time according to eqs. (47) through (50).

A projectile moves in a vertical plane, so its motion can be described using 2D vectors and their components. Let's use a coordinate system where the y axis points vertically up while the x axis is horizontal. (In 3D, the x axis points along the horizontal components of the initial velocity.) In these coordinates, the downward acceleration vector $\vec{a} = \vec{g}$ has components

$$a_x = 0, \quad a_y = -g. \quad (51)$$

Consequently, eqs. (48) and (50) become

$$\begin{aligned} v_x(t) &= v_{0x} = \text{const (time independent)}, \\ x(t) &= x_0 + v_{0x} \times t, \\ v_y(t) &= v_{0y} - g \times t, \\ y(t) &= y_0 + v_{0y} \times t - \frac{g}{2} \times t^2. \end{aligned} \quad (52)$$

Note that the first two of these equations which describe the horizontal motion are completely independent from the last two equations describing the vertical motion. Thus, *the projectile moves horizontally at constant velocity as if there were no vertical motion, and at the same time it moves vertically up and down at constant acceleration as if there were no horizontal motion!*