Bogolyubov Transform

Given some kind of annihilation and creation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ which satisfy the bosonic commutation relations

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0, \quad [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{k,k'}, \quad (1)$$

we may define new operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ according to

$$\hat{b}_k = \cosh(t_k)\hat{a}_k + \sinh(t_k)\hat{a}_k^\dagger, \quad \hat{b}_k^\dagger = \cosh(t_k)\hat{a}_k^\dagger + \sinh(t_k)\hat{a}_{-k} \quad (2)$$

for some arbitrary real parameters $t_k = t_{-k}$. These new operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ satisfy the same bosonic commutation relations as the $\hat{a}_k$ and the $\hat{a}_k^\dagger$:

$$[\hat{b}_k, \hat{b}_{k'}] = [\hat{b}_k^\dagger, \hat{b}_{k'}^\dagger] = 0, \quad [\hat{b}_k, \hat{b}_{k'}^\dagger] = \delta_{k,k'} \quad (3)$$

The Bogolyubov transform — replacing the ‘original’ creation and annihilation operators $\hat{a}_k^\dagger$ and $\hat{a}_k$ with the ‘transformed’ operators $\hat{b}_k^\dagger$ and $\hat{b}_k$ — is useful for diagonalizing quadratic Hamiltonians of the form

$$\hat{H} = \sum_k A_k \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \sum_k B_k \left( \hat{a}_k \hat{a}_{-k} + \hat{a}_k^\dagger \hat{a}_{-k}^\dagger \right) \quad (4)$$

where for all momenta $k$, $A_k = A_{-k}$, $B_k = B_{-k}$, and $A_k > |B_k|$. Indeed, for a suitable choice of the $t_k$ parameters,

$$\hat{H} = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k + \text{const} \quad \text{where} \quad \omega_k = \sqrt{A_k^2 - B_k^2}. \quad (5)$$

Moreover, $\hat{b}_k^\dagger \hat{b}_k - \hat{b}_{-k}^\dagger \hat{b}_{-k} = \hat{a}_k^\dagger \hat{a}_{-k} - \hat{a}_{-k}^\dagger \hat{a}_{-k}$ and consequently

$${\hat{P}} = \sum_k \hat{b}_k^\dagger \hat{b}_k = \sum_k \hat{b}_k^\dagger \hat{b}_k. \quad (6)$$
Proof of (3):
Combining definitions (2) with commutation relations (1), we immediately calculate
\[
[\hat{b}_k, \hat{b}_k'] = \cosh(t_k) \sinh(t_{k'}) \delta_{k, -k'} - \sinh(t_k) \cosh(t_{k'}) \delta_{-k, k'} = 0
\] (7)
where the second equality follows from \( t_k = t_{k'} \) for \( k = -k' \). Likewise, \([\hat{b}_k^\dagger, \hat{b}_k''] = 0 \). Finally,
\[
[\hat{b}_k, \hat{b}_k^\dagger] = \cosh(t_k) \cosh(t_{k'}) \delta_{k, k'} - \sinh(t_k) \sinh(t_{k'}) \delta_{-k, -k'}
\]
\[= \delta_{k, k'} \left( \cosh^2(t_k) - \sinh^2(t_k) = 1 \right). \] (8)

In other words, the \( \hat{b}_k \) and \( \hat{b}_k^\dagger \) operators satisfy the same bosonic commutations relations
\[
[\hat{b}_k, \hat{b}_k^\dagger] = 0, \quad [\hat{b}_k^\dagger, \hat{b}_k'] = 0, \quad [\hat{b}_k, \hat{b}_k'] = \delta_{k, k'}. \] (9)
as the original \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) operators. \( \mathcal{Q.E.D.} \)

Proof of (5):
Applying eqs. (2) twice, we immediately obtain
\[
\hat{b}_k^\dagger \hat{b}_k = \cosh^2(t_k) \hat{a}_k^\dagger \hat{a}_k + \cosh(t_k) \sinh(t_k) (\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k^\dagger \hat{a}_k) + \sinh^2(t_k) (\hat{a}_k^\dagger \hat{a}_k + 1). \] (10)

Next, we use \( t_{-k} = t_k \) to combine
\[
\hat{b}_k^\dagger \hat{b}_k + \hat{b}_{-k}^\dagger \hat{b}_{-k} = \left( \cosh^2(t_k) + \sinh^2(t_k) = \cosh(2t_k) \right) \times (\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k^\dagger \hat{a}_{-k}) \\
+ \left( 2 \cosh(t_k) \sinh(t_k) = \sinh(2t_k) \right) \times (\hat{a}_k^\dagger \hat{a}_{-k} + \hat{a}_{-k}^\dagger \hat{a}_k) + \text{const.} \] (11)

Finally, for \( \omega_{-k} \equiv \omega_k \) we have
\[
\sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k = \frac{1}{2} \sum_k \omega_k (\hat{b}_k^\dagger \hat{b}_k + \hat{b}_{-k}^\dagger \hat{b}_{-k}) \\
= \sum_k \omega_k \cosh(2t_k) \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \sum_k \omega_k \sinh(2t_k) \left( \hat{a}_k^\dagger \hat{a}_{-k} + \hat{a}_{-k}^\dagger \hat{a}_k \right) + \text{const.} \] (12)

Consequently, the Hamiltonian (4) can be “diagonalized” in terms of the transformed creation
annihilation operators (2), provided we can find \( \omega_k = \omega_{-k} \) and \( t_k = t_{-k} \) such that

\[
\omega_k \cosh(2t_k) = A_k \quad \text{and} \quad \omega_k \sinh(2t_k) = B_k.
\]

These equations are easy to solve, and the solution exists as long as \( A_k = A_{-k} \), \( B_k = B_{-k} \), and \( A_k > |B_k| \), namely

\[
t_k = \frac{1}{2} \operatorname{artanh} \frac{B_k}{A_k} \quad \text{and} \quad \omega_k = \sqrt{A_k^2 - B_k^2}.
\]

Q.E.D.

Proof of (6):

Using eq. (10) and \( t_{-k} = t_k \), we immediately see that

\[
\hat{b}_k \hat{b}_k - \hat{b}_{-k} \hat{b}_{-k} = \left( \cosh^2(t_k) - \sinh^2(t_k) = 1 \right) \times (\hat{a}_k \hat{a}_k - \hat{a}_{-k} \hat{a}_{-k}).
\]

Consequently,

\[
\hat{P} = \sum_k \hat{a}_k \hat{a}_k = \sum_k (-\hat{a}_{-k} \hat{a}_{-k})
\]

\[
= \frac{1}{2} \sum_k \left( \hat{a}_k \hat{a}_k - \hat{a}_{-k} \hat{a}_{-k} \right)
\]

\[
= \frac{1}{2} \sum_k \left( \hat{b}_k \hat{b}_k - \hat{b}_{-k} \hat{b}_{-k} \right)
\]

\[
= \sum_k \hat{b}_k \hat{b}_k.
\]

Q.E.D.