1. Consider a theory of $N$ massless Dirac spinor fields $\Psi_1(x), \ldots, \Psi_N(x)$ coupled to the EM massless vector field $A_\mu(x)$. All fermions have the same charge $q = -e$, hence

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \sum_{i=1}^{N} \bar{\Psi}_i (i \varepsilon \not\partial + e A) \Psi_i.$$  \hspace{1cm} (1)

(a) List all symmetries of the classical theory — global or local, discrete or continuous, space-time or internal — and specify how they act on the fields. Skip the time reversal symmetry $T$ and its combination with other symmetries. Also, skip the conformal symmetries.

The rest of this problem focuses on the $U(N)_L \times U(N)_R$ chiral symmetries which act on the Dirac fields according to

$$\Psi'_i(x) = \sum_j \left( \frac{1 - \gamma^5}{2} (U_L)_i^j + \frac{1 + \gamma^5}{2} (U_R)_i^j \right) \Psi_j(x),$$  \hspace{1cm} (2)

where $U_L$ and $U_R$ are two independent $N \times N$ unitary matrices. For infinitesimal chiral symmetries

$$\delta \Psi_i(x) = i \epsilon \times \sum_j \left( \frac{1 - \gamma^5}{2} (T_L)_i^j + \frac{1 + \gamma^5}{2} (T_R)_i^j \right) \Psi_j(x) = i \epsilon \times \sum_j \left( (T_V)_i^j + \gamma^5 (T_A)_i^j \right) \Psi_j,$$  \hspace{1cm} (3)

for hermitian matrices $T_L$ and $T_R$, and hence $T_V = \frac{1}{2} (T_R + T_L)$ and $T_A = \frac{1}{2} (T_R - T_L)$.

(b) Show that the Noether currents of these symmetries are linear combinations of the vector and the axial currents

$$ (V^\mu)_i^j = \bar{\Psi}^i \gamma^\mu \Psi_j \quad \text{and} \quad (A^\mu)_i^j = \bar{\Psi}^i \gamma^\mu \gamma^5 \Psi_j.$$  \hspace{1cm} (4)

Also, verify that all these currents are indeed conserved (in the classical theory).
Now consider the net charge operators in the quantum theory,

\[(\hat{Q}_V)^i_j = \int d^3x \ (\hat{V}^0(x,t))^i_j \] and \[(\hat{Q}_A)^i_j = \int d^3x \ (\hat{A}^0(x,t))^i_j. \] (5)

(c) Calculate the equal-time commutators of these charges with the fermionic fields \(\hat{\Psi}_i(x,t)\) and \(\hat{\bar{\Psi}}^i(x,t)\), then use them to verify that the charges (5) themselves satisfy the commutation relations of the \(U(N) \times U(N)\) generators, namely

\[
\left[\left(\hat{Q}_V\right)^i_j, \left(\hat{Q}_V\right)^k_\ell\right] = \delta^k_j \left(\hat{Q}_V\right)^i_\ell - \delta^k_\ell \left(\hat{Q}_V\right)^i_j, \\
\left[\left(\hat{Q}_V\right)^i_j, \left(\hat{Q}_A\right)^k_\ell\right] = \delta^k_j \left(\hat{Q}_A\right)^i_\ell - \delta^k_\ell \left(\hat{Q}_A\right)^i_j, \\
\left[\left(\hat{Q}_A\right)^i_j, \left(\hat{Q}_A\right)^k_\ell\right] = \delta^k_j \left(\hat{Q}_V\right)^i_\ell - \delta^k_\ell \left(\hat{Q}_V\right)^i_j, \] (6)

(d) Check that all the charges (5) commute with the Hamiltonian operator for the fermionic fields. For simplicity, ignore the EM fields and their interactions with the fermions.

☆ For extra challenge, work with the full Hamiltonian for all the fields. Note that while the transverse components of the EM fields commute with all the fermionic fields (at equal times), the longitudinal electric field is related to fermions by the Gauss law \(\nabla \cdot \hat{E} = J^0_{el} = -e \text{tr} (\hat{V}^0)\).

(e) Finally, expand the charges (5) in terms of fermionic creation and annihilation operators. For simplicity, use the helicity basis for particles’ spin states, and make full use of \(m = 0\). When necessary, subtract an c-number constant due to normal-ordering of fermionic creation and annihilation operators so that the vacuum state has zero net charge.
2. In three spacetime dimensions (two space plus one time) an antisymmetric Lorentz tensor \( F^{\mu\nu} = -F^{\nu\mu} \) is equivalent to an axial Lorentz vector, \( F^{\mu\nu} = \epsilon^{\mu\nu\lambda} F_{\lambda} \). Consequently, in 3D one can make the photons massive without breaking the gauge invariance of the electromagnetic field \( A_\mu(x) \). Indeed, consider the following Lagrangian:

\[
\mathcal{L} = -\frac{1}{2} F_\lambda F^{\lambda} + \frac{m}{2} F_\lambda A^\lambda
\]  

(7)

where

\[
F_\lambda(x) = \frac{1}{2} \epsilon_{\lambda\mu\nu} F^{\mu\nu}(x) = \epsilon_{\lambda\mu\nu} \partial^\mu A^\nu(x),
\]

(8)

or in components, \( F_0 = -B, F_1 = +E^2, F_2 = -E^1 \).

(a) Show that the action \( S = \int d^3 x \mathcal{L} \) is gauge invariant (although the Lagrangian (7) is not invariant).

(b) Write down the classical field equations — including both the Euler–Lagrange equations and the Bianchi identities — for the \( F_\lambda \) fields. Then show that these equations imply the Klein–Gordon equations \((\partial^2 + m^2)F_\lambda(x) = 0\).

Hint: in 2 + 1 dimension \( \epsilon^{\alpha\beta\gamma}\epsilon_\alpha^{\mu\nu} = g^\beta_\mu g^\gamma_\nu - g^\beta_\nu g^\gamma_\mu \).

(c) Write down the plane-wave solutions to the equation of motions and show that for each \( k^\mu = (+\omega, \mathbf{k}) \) there is only one physical polarization that cannot be gauged away. Then argue — but without going through the gory details of quantizing the \( A_\mu(x) \) fields and expanding them into creation and annihilation operators — that the massive photons have only one 2D spin state, either only \( s = +1 \) or only \( s = -1 \), depending on the sign of \( m \).

(d) Write down the Noether stress-energy tensor for the theory in question, then add a suitable total divergence term of the form \( \epsilon^{\mu\alpha\beta} \partial_\alpha K^\nu_\beta \) to make \( T^{\mu\nu} \) gauge invariant and symmetric. The end result should be similar to the stress-energy tensor for the massless EM field,

\[
T^{\mu\nu}_{\text{EM}} = -F^{\mu\lambda} F^\nu_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} = +F^\mu F^\nu - \frac{1}{2} g^{\mu\nu} F_\lambda F^\lambda \quad \text{(in 3D)}.
\]

(9)
3. Finally, consider the quantum theory of the massive EM fields in 3D. The Hamiltonian operator follows from the classical net energy $\int d^2x T^{0,0}$, thus in light of eq. (9)

$$\hat{H} = \int d^2x \left( \hat{T}^{0,0} = \frac{1}{2}(\hat{F}_1)^2 + \frac{1}{2}(\hat{F}_2)^2 + \frac{1}{2}(\hat{F}_0)^2 \right). \quad (10)$$

Note that this Hamiltonian is independent on the mass $m$. Instead, the mass affects the equal-time commutation relations of the quantum fields.

(a) Write down the canonical conjugate fields $\Pi^{1,2}(x)$ to the vector potentials $A^{1,2}(x)$, re-express the electric fields $F^{1,2}$ in terms of the $\Pi^{1,2}$ and $A^{1,2}$, then show that the canonical commutation relations between the quantum $\Pi^{1,2}$ and $A^{1,2}$ fields lead to

$$\left[ \hat{F}_1(x, t), \hat{F}_2(y, \text{same } t) \right] = -im \delta^{(2)}(x - y). \quad (11)$$

(b) One of the classical equations of motion for the $F_\mu(x, t)$ fields is time-independent (i.e., does not involve time derivatives). Impose it as an operatorial identity expressing the quantum $\hat{F}_0(x, t)$ field in terms of space derivatives of the $\hat{F}_1(x, t)$ and $\hat{F}_2(x, t)$, then use it to derive the equal-time commutations relations between the $\hat{F}_0(x, t)$ and the $\hat{F}_{1,2}(y, t)$.

(c) Finally, use all these commutation relations and the Hamiltonian (10) to show that in the Heisenberg picture, the quantum $\hat{F}_\lambda(x)$ fields obey similar equations of motion to the classical fields.