Non–Abelian Higgs Mechanism

When a local rather than global symmetry is spontaneously broken, we do not get a massless Goldstone boson. Instead, the gauge field of the broken symmetry becomes massive, and the would-be Goldstone scalar becomes the longitudinal mode of the massive vector. This is the \textit{Higgs mechanism}, and it works for both abelian and non-abelian local symmetries. In the non-abelian case, for each spontaneously broken generator $T^a$ of the local symmetry the corresponding gauge field $A^a_\mu(x)$ becomes massive.

\textbf{Example: SU(2) with a Higgs Doublet}

To illustrate the non-abelian Higgs mechanism, consider the example of SU(2) gauge theory coupled to a doublet of complex scalar fields $\Phi_i(x)$. In terms of canonically normalized fields, the Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu} + D_\mu \Phi^* i D^\mu \Phi_i - \frac{\lambda}{2} \left( \Phi^* i \Phi_i - \frac{v^2}{2} \right)^2,$$

where

$$D_\mu \Phi_i = \partial_\mu \Phi_i + i g A^a_\mu (\sigma^a)_{ij} \Phi_j,$$
$$D_\mu \Phi^* i = \partial_\mu \Phi^* i - i g A^a_\mu \Phi^* j (\sigma^a)_{ij},$$
$$F^{a}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g e^{abc} A^b_\mu A^c_\nu.$$  

For $v^2 > 0$ the scalar potential has a local maximum at $\Phi_i = 0$ while the minima form a spherical shell $\Phi^* i \Phi_i = (v^2/2)$ in the $C^2 = R^4$ field space; all such minima are related by SU(2) symmetries to

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Note that this vacuum expectation value spontaneously breaks the SU(2) symmetry down to nothing — there is no subgroup of SU(2) which leaves this VEV invariant. Consequently, we expect all 3 vector fields $A^a_\mu(x)$ to become massive.

In the process, 3 would-be Goldstone scalars should be eaten by the Higgs mechanism. Since the theory has 2 complex — or equivalently 4 real — scalars, only one real scalar should survive un-eaten. Ironically, it is this un-eaten scalar $\sigma(x)$ which is called \textit{the physical Higgs field}. 

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To see how this works, let’s fix the unitary gauge where

\[
\text{Re } \Phi_1(x) \equiv \text{Im } \Phi_1(x) \equiv \text{Im } \Phi_2(x) \equiv 0,
\]

\[
\Phi(x) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ \phi_r(x) \end{array} \right), \quad \text{real } \phi_r(x) > 0,
\]

and then shift the \( \phi_r \) field by the VEV, \( \phi_r(x) = v + \sigma(x) \). For \( v = 0 \) such gauge fixing would be terribly singular, but it’s perfectly OK for \( v \neq 0 \) and \( |\sigma(x)| < v \implies \phi_r(x) \neq 0 \).

In the unitary gauge, the physical Higgs field \( \sigma(x) \) is the only scalar field, the rest are frozen by the gauge-fixing conditions (4). In terms of \( \sigma \), the scalar potential becomes

\[
V = \frac{\lambda}{2} \left( \Phi^\dagger \Phi - \frac{v^2}{2} \right)^2 = \frac{\lambda}{8} (2v\sigma + \sigma^2)^2 = \frac{\lambda v^2}{2} \times \sigma^2 + \frac{\lambda v}{2} \times \sigma^3 + \frac{\lambda}{8} \times \sigma^4
\]

(5)

where the first terms is the mass term, mass\(^2 = \lambda v^2 \), while the remaining terms are self-interactions. More interestingly, the covariant derivative of the Higgs doublet \( \Phi \) becomes

\[
D_\mu \Phi = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} 0 \\ \partial_\mu \sigma \end{array} \right] + \frac{ig}{2} A_3^\mu \times \left( \begin{array}{cc} +1 & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{c} 0 \\ v + \sigma \end{array} \right) \\
+ \frac{ig}{2} A_1^\mu \times \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ v + \sigma \end{array} \right) + \frac{ig}{2} A_2^\mu \times \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ v + \sigma \end{array} \right)
\]

\[
= \frac{1}{\sqrt{2}} \left( \frac{ig}{2} (A_1^\mu - iA_2^\mu) \times (v + \sigma) \right) + \frac{ig}{2} A_3^\mu \times (v + \sigma),
\]

(6)

hence

\[
D_\mu \Phi^\dagger D^\mu \Phi = \frac{1}{2} \left| \frac{ig}{2} (A_1^\mu - iA_2^\mu) \times (v + \sigma) \right|^2 + \frac{1}{2} \left| \partial_\mu \sigma - \frac{ig}{2} A_3^\mu \times (v + \sigma) \right|^2
\]

\[
= \frac{g^2 (v + \sigma)^2}{8} \times (A_1^\mu)^2 + (A_2^\mu)^2 \right) + \frac{g^2 (v + \sigma)^2}{8} \times (A_3^\mu)^2 + \frac{1}{2} (\partial_\mu \sigma)^2.
\]

(7)

The last term here is the kinetic term for the Higgs scalar \( \sigma(x) \), while the rest of the bottom line are mass terms for the vector fields and the interaction terms between the vectors and
the $\sigma$. Curiously, we get the same mass and similar interactions for all 3 vector fields $A^a_\mu$:

$$\mathcal{L} \supset \frac{g^2(v + \sigma)^2}{8} A^a_\mu A^{a\mu} = \frac{M^2}{2} A^a_\mu A^{a\mu} + \frac{g^2 v}{4} \sigma A^a_\mu A^{a\mu} + \frac{g^2}{8} \sigma^2 A^a_\mu A^{a\mu}$$  \hspace{1cm} (8)

where

$$M^2 = \frac{g^2 v^2}{4}.$$  \hspace{1cm} (9)

**Example: SU(2) with a Higgs Triplet**

Now consider an example of a partially broken gauge symmetry, $SU(2)$ Higgsed down to a $U(1)$ subgroup, or equivalently $SO(3) \rightarrow SO(2)$. This time, the scalar fields $\Phi^a(x)$ are real and form a triplet of the $SU(2)$ rather than a doublet. Thus,

$$\mathcal{L} = -\frac{1}{4} F^{a\mu\nu} F^{a\mu\nu} + \frac{1}{2} D^a_\mu \Phi^a D^a_\mu \Phi^a - \frac{\lambda}{8} (\Phi^a \Phi^a - v^2)^2,$$  \hspace{1cm} (10)

where

$$D^a_\mu \Phi^a = \partial^a_\mu \Phi^a - g^{abc} A^b_\mu \Phi^c, \quad F^{a\mu\nu} = \partial^a_\mu A^a_\nu - \partial^a_\nu A^a_\mu - g^{abc} A^b_\mu A^c_\nu.$$  \hspace{1cm} (11)

Again, for $v^2 > 0$ the scalar potential $V(\Phi)$ has a degenerate family of minima which form a spherical shell $\Phi^a \Phi^a = v^2$ in the scalar field space $\mathbb{R}^3$, and all such minima are equivalent by $SU(2) \cong SO(3)$ symmetries to

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}.$$  \hspace{1cm} (12)

This time, this vacuum expectation value is invariant under the $SO(2)$ subgroup of $SO(3)$, — or equivalently under the $U(1)$ subgroup of $SU(2)$ — generated by the $T^3$ (the third component of the isospin $T$). Consequently, out of 3 vector fields $A^a_\mu$, we expect the $A^3_\mu$ to remain massless while the other 2 fields $A^{1,2}_\mu$ should become massive.

In the process, the Higgs mechanism should eat 2 real scalar fields. Since we only have 3 real scalars to begin with, only one scalar should survive un-eaten — the Physical Higgs field $\sigma(x)$. 

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To see how this works, we fix the unitary gauge

$$\Phi^1(x) \equiv \Phi^2(x) \equiv 0, \quad \Phi^3(x) > 0. \quad (13)$$

As usual, this gauge is badly singular for $\Phi = 0$, but it’s OK for $\Phi(x)$ being close to the VEV $\langle \Phi \rangle \neq 0$. Shifting $\Phi^3(x)$ by the VEV, we get $\Phi^3(x) = v + \sigma(x)$, where $\sigma(x)$ is the physical Higgs scalar — and also the only scalar remaining in the theory in the unitary gauge.

In terms of $\sigma(x)$, the scalar potential becomes

$$V(\sigma) = \frac{\lambda}{8} (2v\sigma + \sigma^2)^2 = \frac{\lambda v^2}{2} \times \sigma^2 + \frac{\lambda v^2}{2} \times \sigma^3 + \frac{\lambda}{8} \times \sigma^4, \quad (14)$$

where the first terms on the RHS gives the Higgs scalar mass $^2 = \lambda v^2$. More interestingly, the covariant derivative of the scalar triple $\Phi^a(x)$ becomes

$$D_\mu \Phi^a = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - g \begin{pmatrix} A^1_\mu \\ A^2_\mu \\ A^3_\mu \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ v + \sigma \end{pmatrix}$$

$$\langle \text{where } \times \text{ is the cross product of two isovectors} \rangle \quad (15)$$

$$= \begin{pmatrix} -gA^2_\mu(v + \sigma) \\ +gA^1_\mu(v + \sigma) \\ \partial_\mu \sigma \end{pmatrix},$$

hence the covariant kinetic terms for the scalars become

$$\frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a = \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{g^2(v + \sigma)^2}{2} \times (A^1_\mu)^2 + (A^2_\mu)^2. \quad (16)$$

As usual, the first term here is the kinetic term for the physical Higgs scalar $\sigma$, while the second term contains mass terms

$$\frac{M^2}{2} \times \left( (A^1_\mu)^2 + (A^2_\mu)^2 \right), \quad M^2 = g^2 v^2, \quad (17)$$

for the vector fields, but only for the $A^1_\mu$ and $A^2_\mu$ — the third vector $A^3_\mu(x)$ remains massless.
The massless vector $A_\mu^3(x)$ is the gauge field of the un-Higgsed $SO(2) \cong U(1)$ subgroup of the $SO(3) \cong SU(2)$. Interpreting the generator $Q = gT^3$ of this subgroup as electric charge, we find that the massive vector fields, or rather their linear combinations

$$ W^+_{\mu} = \frac{1}{\sqrt{2}} (A^1_{\mu} + iA^2_{\mu}) \quad \text{and} \quad W^-_{\mu} = \frac{1}{\sqrt{2}} (A^1_{\mu} - iA^2_{\mu}) $$

have charges $\pm g$ while the physical Higgs field $\sigma$ is neutral.

For completeness sake, let’s re-express the theory at hand (usually called the Georgi–Glashow model) in terms of physical fields of definite charges. Using $U(1)$–covariant derivatives

$$ \tilde{D}_\mu W^\pm_\nu = \partial_\mu W^\pm_\nu \pm igA^3_\mu W^\pm_\nu, \quad (19) $$

we have

$$ W^\pm_{\mu\nu} \overset{\text{def}}{=} \frac{1}{\sqrt{2}} (F^1_{\mu\nu} \pm iF^\pm_{\mu\nu}) = \tilde{D}_\mu W^\pm_\nu - \tilde{D}_\nu W^\pm_\mu, \quad (20) $$

but

$$ F^3_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu + 2g \text{Im}(W^+_\mu W^-_\nu). \quad (21) $$

Consequently, the Lagrangian of the whole model — the kinetic terms, the mass terms, and the interactions — can be expressed as

$$ \mathcal{L} = \frac{1}{2} (\partial_\mu \sigma)^2 - V(\sigma) $$

$$ - \frac{1}{4} \left( \partial_\mu A^3_\nu - \partial_\nu A^3_\mu + 2g \text{Im}(W^+_\mu W^-_\nu) \right)^2 $$

$$ - \frac{1}{2} W^+_\mu W^-_\mu + (M + g\sigma)^2 \times W^+_\mu W^-_\mu. \quad (22) $$
**General Case**

Let’s take a closer look at eqs. (7) and (16), and focus on the mass terms for the vector fields. In both cases, we start with the kinetic terms for the original scalar fields \( \Phi_i(x) \) or \( \Phi^a(x) \), fix the unitary gauge, work through the algebra, and eventually obtain the kinetic term for the physical Higgs field \( \sigma \), the mass terms for the vector fields — or some of the vector fields — and the interactions between the massive vectors and the Higgs \( \sigma \). But is all we want are the mass terms for the vectors, we may simply freeze \( \sigma(x) \equiv 0 \): This would eliminate the interactions with the \( \sigma \) as well as the \( \frac{1}{2}(\partial_\mu \sigma) \) term, and all we would have left are the mass terms for the massive vectors.

Note that freezing \( \sigma(x) \equiv 0 \) is equivalent to freezing all the scalars at their VEVs, \( \Phi(x) \equiv \langle \Phi \rangle \). Consequently, to get the vector’s masses we do not need to go through the details of the unitary gauge fixing, all we need are the scalar VEVs, then the kinetic terms for the frozen scalars

\[
D_\mu \langle \Phi \rangle |^i D_\mu \langle \Phi \rangle \quad \text{or} \quad \frac{1}{2} (D_\mu \langle \Phi \rangle)^2
\]

becomes the mass terms for the vectors. For example, for the \( SO(3) \) triplet of real scalar fields from the second example

\[
D_\mu \langle \Phi \rangle^a = -g \epsilon^{abc} A^b_\mu \times v \delta^c 3 = -g v \epsilon^{ab3} \times A^b_\mu, \quad (23)
\]

\[
\mathcal{L}_\text{vector mass} = \frac{1}{2} (D_\mu \langle \Phi \rangle^a)^2 = \frac{1}{2} (gv)^2 \times \epsilon^{ab3} \epsilon^{ac3} A^b_\mu A^c_\mu
\]

\[
= \frac{1}{2} (M = gv)^2 \times (A^1_\mu A^1_\mu + A^2_\mu A^2_\mu). \quad (24)
\]

Likewise, for the \( SU(2) \) doublet of complex scalar fields from the first example,

\[
D_\mu \langle \Phi \rangle^i = \frac{ig}{2} (A_\mu^a \sigma^a)^j_i \times \frac{v}{\sqrt{2}} \delta^i_j = \frac{igv}{4} \times (A_\mu^a \sigma^a)^i_i, \quad (25)
\]

\[
D_\mu \langle \Phi \rangle^{*i} = -\frac{igv}{2 \sqrt{2}} \times (A_\mu^a \sigma^a)^i, \quad (26)
\]

\[
\mathcal{L}_\text{vector mass} = D_\mu \langle \Phi \rangle^{*i} D^\mu \langle \Phi \rangle^i = \frac{g^2 v^2}{8} \times (A_\mu^a \sigma^a)^i (A^b_\mu \sigma^b)^i
\]

\[
= \frac{g^2 v^2}{8} \times A_\mu^a A^b_\mu \times \left[ (\sigma^a \sigma^b)^i \right] = \delta^{ab} - i \epsilon^{ab3}
\]
\[
\begin{align*}
&= \frac{g^2 v^2}{8} \times A^a_{\mu} A^{b\mu} \times \delta^{ab} \quad \langle\text{since } A^a_{\mu} A^{b\mu} \text{ is symmetric in } a \leftrightarrow b.\rangle \\
&= \frac{M^2}{2} \times A^a_{\mu} A^{a\mu} \quad \text{for } M = \frac{g v}{2}.
\end{align*}
\] (27)

This recipe — freezing \( \Phi(x) \equiv \langle \Phi \rangle \) to find the vector masses — applies to any kind of gauge theory with scalars in any kinds of multiplets. Indeed, consider a general gauge symmetry \( G \) with generators \( \hat{T}^a \) and gauge fields \( A^a_{\mu}(x) \) \( (a = 1, \ldots, \dim(G)). \) Let scalars \( \Phi_{\alpha}(x) \) belonging to some multiplet \( (m) \) of \( G \) develop non-zero vacuum expectation values \( \langle \Phi_{\alpha} \rangle \neq 0. \) Then the covariant derivatives of these scalars

\[
D_{\mu} \Phi_{\alpha}(x) = \partial_{\mu} \Phi_{\alpha}(x) + ig A^a_{\mu}(x) \times (T^a_{(m)})_{\alpha}^\beta \Phi_{\beta}(x)
\] (28)

become in the unitary gauge

\[
D_{\mu} \Phi_{\alpha}(x) = D_{\mu} \langle \Phi \rangle_{\alpha} + \text{terms involving the physical scalars}
\] (29)

where

\[
D_{\mu} \langle \Phi \rangle = ig A^a_{\mu}(x) \times (T^a_{(m)})_{\alpha}^\beta \langle \Phi \rangle_{\beta}.
\] (30)

In eq. (29), the terms involving the physical scalars — and the physical scalar fields themselves — depend on the details of the unitary gauge fixing. On the other hand, the covariant derivatives of the VEVs (30) depend only on the VEVs themselves. Moreover, such derivatives are linear functions of the vector fields with constant coefficients, so their squares become quadratic mass terms for the vectors,

\[
D^\mu \langle \Phi \rangle^{*\alpha} D_{\mu} \langle \Phi \rangle_{\alpha} = -ig A^a_{\mu} \times \langle \Phi \rangle^{*\beta} (T^a_{(m)})_{\beta}^\alpha \times ig A^{b\mu} \times (T^a_{(m)})_{\alpha}^\gamma \langle \Phi \rangle_{\gamma}
\]
\[
= A^a_{\mu} A^{b\mu} \times g^2 \langle \Phi \rangle^{*\beta} (T^a_{(m)} T^b_{(m)})_{\beta}^\gamma \langle \Phi \rangle_{\gamma} \quad \langle\text{by } a \leftrightarrow b \text{ symmetry of the } A^a_{\mu} A^{b\mu}\rangle
\]
\[
= \frac{1}{2} A^a_{\mu} A^{b\mu} \times g^2 \langle \Phi \rangle^{*\beta} \{ T^a_{(m)} T^b_{(m)} \}_{\beta}^\gamma \langle \Phi \rangle_{\gamma}.
\] (31)

In other words, we get the mass\(^2\) matrix for the gauge fields

\[
\mathcal{L}_{\text{masses}}^{\text{vector}} = \frac{1}{2} (M^2_{\gamma})^{ab} \times A^a_{\mu} A^{b\mu}, \quad \text{where}
\] (32)
\[
(M^2_V)^{ab} = g^2 \langle \Phi \rangle^* \{ T^a_{(m)}, T^b_{(m)} \}_\beta^\gamma \langle \Phi \rangle^\gamma \equiv g^2 \langle \Phi \rangle^\dagger \{ T^a_{(m)}, T^b_{(m)} \} \langle \Phi \rangle .
\]

(33)

To be precise, eq. (33) applies to Higgs VEVs belonging to a single multiplet of complex scalars. For a multiplet of real scalars, there is an extra factor \( \frac{1}{2} \) due to different normalization of the VEVs, and for several Higgs multiplets with non-zero VEVs, the general formula is

\[
(M^2_V)^{ab} = g^2 \sum_{\Phi \in (m)} \langle \Phi \rangle^\dagger \{ T^a_{(m)}, T^b_{(m)} \} \langle \Phi \rangle + g^2 \sum_{\Phi \in (m)} \frac{1}{2} \langle \Phi \rangle^\top \{ T^a_{(m)}, T^b_{(m)} \} \langle \Phi \rangle .
\]

(34)

In general, such mass\(^2\) matrix is not diagonal, and we need to diagonalize in order to find the physical vector masses. For example, in the Glashow–Weinberg–Salam theory of the weak and EM interactions — it’s explained in the next set of notes — the mass matrix mixes an \( SU(2) \) gauge field \( W^3_\mu \) and the \( U(1) \) gauge field \( B_\mu \), and the mass eigenstates are the massless EM field \( A^\mu \) and the massive neural field \( Z^\mu \) involved in the weak interactions.

An additional complication of the GWS theory — or any other theory with non-simple gauge group \( G = G_1 \times G_2 \times \cdots \) — are different gauge couplings \( g \) for different factors \( G \). In this case, the \( g^2 \) factor in eq. (34) for the mass\(^2\) matrix element \((M^2)^{ab}\) should be replaced with \( g(a) \times g(b) \) where \( g(a) \) is the coupling of the gauge group factor containing the generator \( T^a \), and likewise for the \( g(b) \). Thus, the most general formula for the vector mass matrix stemming from the Higgs mechanism is

\[
(M^2_V)^{ab} = g(a) g(b) \times \left[ \sum_{\Phi \in (m)} \langle \Phi \rangle^\dagger \{ T^a_{(m)}, T^b_{(m)} \} \langle \Phi \rangle + \frac{1}{2} \sum_{\Phi \in (m)} \langle \Phi \rangle^\top \{ T^a_{(m)}, T^b_{(m)} \} \langle \Phi \rangle \right] .
\]

(35)

In my notes on the GWS theory we shall see how this works in detail, and how the gauge couplings affect the eigenstates of the mass matrix.