Mandelstam Variables

Consider any kind of a 2 particles → 2 particles process

\[ 1' \rightarrow 2' \] \hspace{1cm} (1)

The 4-momenta \( p_1^\mu \), \( p_2^\mu \), \( p_1'^\mu \), and \( p_2'^\mu \) of the 2 incoming and 2 outgoing particles satisfy 8 constraints: the on-shell conditions for each particle

\[ p_1^2 = m_1^2, \quad p_2^2 = m_2^2, \quad p_1'^2 = m_1'^2, \quad p_2'^2 = m_2'^2, \] \hspace{1cm} (2)

and the net 4-momentum conservation

\[ p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu. \] \hspace{1cm} (3)

Altogether, this gives us \( 4 \times 4 - 8 = 8 \) independent momentum variables, and the number of independent Lorentz-invariant combinations of these variables is only \( 8 - 6 = 2 \).

However, for practical purposes it’s is often convenient to use 3 Lorentz-invariant variables with a fixed sum,

\[ s = (p_1 + p_2)^2 = (p_1' + p_2')^2, \]
\[ t = (p_1 - p_1')^2 = (p_2' - p_2)^2, \]
\[ u = (p_1 - p_2')^2 = (p_1' - p_2)^2. \] \hspace{1cm} (4)

Indeed,

\[ s + t + u = (p_1 + p_2)^2 + (p_1 - p_1')^2 + (p_1 - p_2')^2 = 3p_1^2 + p_2^2 + p_1'^2 + p_2'^2 + 2(p_1p_2) - 2(p_1p_1') - 2(p_1p_2') \]
\[ = p_1^2 + p_2^2 + p_1'^2 + p_2'^2 + 2p_1 \times (p_1 + p_2 - p_1' - p_2' = 0) \]
\[ = p_1^2 + p_2^2 + p_1'^2 + p_2'^2 \]
\[ = m_1^2 + m_2^2 + m_1'^2 + m_2'^2. \] \hspace{1cm} (5)

The \( s \), \( t \), and \( u \) are called Mandelstam variables after Stanley Mandelstam who introduced them back in 1958.
All Lorentz-invariant combinations of the four momenta $p_1^\mu$, $p_2^\mu$, $p_1'^\mu$, and $p_2'^\mu$ can be expressed in terms of the Mandelstam variables. For example, the Lorentz products $k^\mu k'_\mu$ of any two momenta are

$$2(p_1p_2) = (p_1 + p_2)^2 - p_1^2 - p_2^2 = s - m_1^2 - m_2^2,$$

$$2(p_1'p_2') = (p_1' + p_2')^2 - p_1'^2 - p_2'^2 = s - m_1'^2 - m_2'^2,$$

$$2(p_1p_1') = p_1^2 + p_1'^2 - (p_1 - p_1')^2 = m_1^2 + m_1'^2 - t,$$

$$2(p_2p_2') = p_2^2 + p_2'^2 - (p_2 - p_2')^2 = m_2^2 + m_2'^2 - t,$$

$$2(p_1p_1') = p_1^2 + p_1'^2 - (p_1 - p_1')^2 = m_1^2 + m_2'^2 - u,$$

$$2(p_2p_2') = p_2^2 + p_2'^2 - (p_2 - p_1')^2 = m_2^2 + m_1'^2 - u.$$  \hspace{1cm} (6)

In particular, for an elastic scattering of 2 same-mass particles

$$s + t + u = 4m^2,$$

$$2(p_1p_2) = 2(p_1'p_2') = s - 2m^2,$$

$$2(p_1p_1') = 2(p_2p_2') = 2m^2 - t,$$

$$2(p_2p_2') = 2(p_2p_1') = 2m^2 - u.$$ \hspace{1cm} (7)

For future reference, let me give you similar formulae for the $e^- e^+ \rightarrow \mu^- \mu^+$ pair-production,

$$s + t + u = 2M_\mu^2 + 2m_e^2 \approx 2M_\mu^2,$$

$$2(p_1p_2) = s - 2m_e^2 \approx s,$$

$$2(p_1'p_2') = s - 2M_\mu^2,$$ \hspace{1cm} (8)

$$2(p_1p_1') = 2(p_2p_2') = M_\mu^2 + m_e^2 - t \approx M_\mu^2 - t,$$

$$2(p_1p_2') = 2(p_2p_1') = M_\mu^2 + m_e^2 - u \approx M_\mu^2 - u.$$

and for the $e^- e^+ \rightarrow \gamma \gamma$ annihilation process $p_- + p_+ \rightarrow k_1 + k_2$,

$$s + t + u = 2m_e^2,$$

$$2(p_-p_+) = s - 2m_e^2,$$

$$2(k_1k_2) = s,$$

$$2(p_-k_1) = 2(p_+k_2) = m_e^2 - t,$$

$$2(p_-k_2) = 2(p_+k_1) = m_e^2 - u.$$ \hspace{1cm} (9)