Poisson Brackets and Commutator Brackets

Both classical mechanics and quantum mechanics use bi-linear brackets of variables with similar algebraic properties. In classical mechanics the variables are functions of the canonical coordinates and momenta, and the Poisson bracket of two such variables $A(q,p)$ and $B(q,p)$ are defined as

$$[A, B]_P \overset{\text{def}}{=} \sum_i \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right). \quad (1)$$

In quantum mechanics the variables are linear operators in some Hilbert space, and the commutator bracket of two operators is

$$[A, B]_C \overset{\text{def}}{=} AB - BA. \quad (2)$$

Both types of brackets have similar algebraic properties:

1. **Linearity:** $[\alpha_1 A_1 + \alpha_2 A_2, B] = \alpha_1 [A_1, B] + \alpha_2 [A_2, B]$ and $[A, \beta_1 B_1 + \beta_2 B_2] = \beta_1 [A, B_1] + \beta_2 [A, B_2]$.


**Theorem:** For non-commuting variables, any bracket $[A, B]$ with the above algebraic properties 1 through 4 is proportional to the commutator bracket:

$$[A, B] = c(AB - BA) \quad (3)$$

for a universal constant $c$ (same $c$ for all variables). In particular, generalization of classical Poisson brackets to quantum mechanics leads to

$$[\hat{A}, \hat{B}]_P = \frac{\hat{A}\hat{B} - \hat{B}\hat{A}}{\imath \hbar} \quad (4)$$
**Proof:** Take any 4 variables $A, B, U, V$ and calculate $[AU, BV]$ using the Leibniz rules, first for the $AU$ and then for the $BV$:

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OOH, if we use the two Leibniz rules in the opposite order we get a different expression

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To make sure the two expressions are equal to each other we need

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\begin{align*}
AB[U, V] + [A, B]VU &= BA[U, V] + [A, B]UV \\
\implies (AB - BA)[U, V] &= [A, B](UV - VU) \tag{7} \\
[U, V](UV - VU)^{-1} &= (AB - BA)^{-1}[A, B]
\end{align*}
\]

On the last line here, the LHS depends only on the $U$ and $V$ while the RHS depends only on the $A$ and $B$, and the only way a relation like that can work for any unrelated variables is if the ratios on both sides of equations are equal to the same universal constant $c$, thus

\[
[A, B] = c(AB - BA) \quad \text{and} \quad [U, V] = c(UV - VU). \tag{8}
\]

*Quod erat demonstrandum.*