1. First, an exercise in bosonic commutation relations

\[ [\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] = 0, \quad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta}. \quad (1) \]

(a) Calculate the commutators \([\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta], [\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta], [\hat{a}_\alpha \hat{a}^\dagger_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta], \text{ and } [\hat{a}_\alpha \hat{a}^\dagger_\beta \hat{a}^\dagger_\gamma \hat{a}_\delta, \hat{a}^\dagger_\mu \hat{a}_\nu].\]

(b) For a single pair of \(\hat{a}\) and \(\hat{a}^\dagger\) operators, show that for any analytic function \(f(x) = f_0 + f_1 x + f_2 x^2 + \cdots\),

\[ [\hat{a}, f(\hat{a}^\dagger)] = +f'(\hat{a}^\dagger) \quad \text{and} \quad [\hat{a}^\dagger, f(\hat{a})] = -f'(\hat{a}) \quad (2) \]

where \(f(\hat{a}) \overset{\text{def}}{=} f_0 + f_1 \hat{a} + f_2 (\hat{a})^2 + \cdots\) and likewise \(f(\hat{a}^\dagger) \overset{\text{def}}{=} f_0 + f_1 \hat{a}^\dagger + f_2 (\hat{a}^\dagger)^2 + \cdots\).

(c) Show that \(e^{c\hat{a}^\dagger} \hat{a} e^{-c\hat{a}} = \hat{a}^\dagger + c, e^{c\hat{a}^\dagger} \hat{a} e^{-c\hat{a}^\dagger} = \hat{a} - c\), hence for any analytic function \(f\),

\[ e^{c\hat{a}^\dagger} f(\hat{a}^\dagger) e^{-c\hat{a}} = f(\hat{a}^\dagger + c) \quad \text{and} \quad e^{c\hat{a}^\dagger} f(\hat{a}) e^{-c\hat{a}^\dagger} = f(\hat{a} - c). \quad (3) \]

(d) Now generalize (b) and (c) to any set of creation and annihilation operators \(\hat{a}_\alpha^\dagger\) and \(\hat{a}_\alpha\). Show that for any analytic function \(f(\text{multiple } \hat{a}_\alpha^\dagger)\) of creation operators but not of the annihilation operators or a function \(f(\text{multiple } \hat{a}_\alpha)\) of the annihilation operators but not of the creation operators,

\[ [\hat{a}_\alpha, f(\hat{a}^\dagger)] = +\frac{\partial f(\hat{a}^\dagger)}{\partial \hat{a}_\alpha^\dagger}, \quad [\hat{a}_\alpha^\dagger, f(\hat{a})] = -\frac{\partial f(\hat{a})}{\partial \hat{a}_\alpha}, \quad (4) \]

\[ \exp \left( \sum_\alpha c_\alpha \hat{a}_\alpha \right) f(\hat{a}^\dagger) \exp \left( -\sum_\alpha c_\alpha \hat{a}_\alpha \right) = f(\text{each } \hat{a}_\alpha^\dagger \rightarrow \hat{a}_\alpha^\dagger + c_\alpha), \]

\[ \exp \left( \sum_\alpha c_\alpha \hat{a}_\alpha^\dagger \right) f(\hat{a}) \exp \left( -\sum_\alpha c_\alpha \hat{a}_\alpha^\dagger \right) = f(\text{each } \hat{a}_\alpha \rightarrow \hat{a}_\alpha - c_\alpha). \]
2. Now consider an $O(N)$ symmetric Lagrangian for $N$ interacting real scalar fields,

$$L = \frac{1}{2} \sum_{a=1}^{N} (\partial_{\mu} \Phi_a)^2 - \frac{m^2}{2} \sum_{a=1}^{N} \Phi_a^2 - \frac{\lambda}{24} \left( \sum_{a=1}^{N} \Phi_a^2 \right)^2. \quad (5)$$

By the Noether theorem, the continuous $SO(N)$ subgroup of the $O(N)$ symmetry gives rise to $\frac{1}{2} N(N-1)$ conserved currents

$$J_{\mu}^{ab}(x) = -J_{\mu}^{ba}(x) = \Phi_a(x) \partial_{\mu} \Phi_b(x) - \Phi_b(x) \partial_{\mu} \Phi_a(x). \quad (6)$$

In the quantum field theory, these currents become operators

$$\hat{J}_{\mu}^{ab}(x,t) = -\hat{J}_{\mu}^{ba}(x,t) = \hat{\Phi}_a(x,t) \nabla \hat{\Phi}_b(x,t) + \hat{\Phi}_b(x,t) \nabla \hat{\Phi}_a(x,t),$$

$$\hat{J}_0^{ab}(x,t) = -\hat{J}_0^{ba}(x,t) = \hat{\Phi}_a(x,t) \hat{\Pi}_b(x,t) - \hat{\Phi}_b(x,t) \hat{\Pi}_a(x,t). \quad (7)$$

This problem is about the net charge operators

$$\hat{Q}_{ab}(t) = -\hat{Q}_{ba}(t) = \int d^3x \hat{J}_0^{ab}(x) = \int d^3x \left( \hat{\Phi}_a(x,t) \hat{\Pi}_b(x,t) - \hat{\Phi}_b(x,t) \hat{\Pi}_a(x,t) \right). \quad (8)$$

(a) Write down the equal-time commutation relations for the quantum $\hat{\Phi}_a$ and $\hat{\Pi}_a$ fields.

Also, write down the Hamiltonian operator for the interacting fields.

(b) Show that

$$\left[ \hat{Q}_{ab}(t), \Phi_c(x, \text{same } t) \right] = i \delta_{bc} \hat{\Phi}_a(x,t) - i \delta_{ac} \hat{\Phi}_b(x,t),$$

$$\left[ \hat{Q}_{ab}(t), \Pi_c(x, \text{same } t) \right] = i \delta_{bc} \hat{\Pi}_a(x,t) - i \delta_{ac} \hat{\Pi}_b(x,t), \quad (9)$$

(c) Show that the all the $\hat{Q}_{ab}$ commute with the Hamiltonian operator $\hat{H}$. In the Heisenberg picture, this makes all the charge operators $\hat{Q}_{ab}$ time independent.

(d) Verify that the $\hat{Q}_{ab}$ obey commutation relations of the $SO(N)$ generators,

$$\left[ \hat{Q}_{ab}, \hat{Q}_{cd} \right] = -i \delta_{[c|b]} \hat{Q}_{a|d]} \equiv -i \delta_{bc} \hat{Q}_{ad} + i \delta_{ac} \hat{Q}_{bd} + i \delta_{bd} \hat{Q}_{ac} - i \delta_{ad} \hat{Q}_{bc}. \quad (10)$$

Now let’s take $\lambda \rightarrow 0$ and focus on the free fields. Let’s work in the Schrödinger picture and expand all the fields into creation and annihilation operators $\hat{a}^\dagger_{p,a}$ and $\hat{a}_{p,a}$ ($a = 1, \ldots, N$).
(e) Show that in terms of creation and annihilation operators, the charges (8) become

\[ \hat{Q}_{ab} = \sum_p \left( -i \hat{a}_{p,a}^\dagger \hat{a}_{p,b} + i \hat{a}_{p,b}^\dagger \hat{a}_{p,a} \right). \]  

(11)

(f) Use the commutation relations (1) for the creation and annihilation operators (and the results of problem 1.a) to verify that the operators (11) obey the commutation relations (10).

Finally, for \( N = 2 \) the \( SO(2) \) symmetry is the phase symmetry of one complex field \( \Phi = (\Phi_1 + i\Phi_2)/\sqrt{2} \) and its conjugate \( \Phi^* = (\Phi_1 - i\Phi_2)/\sqrt{2} \). In the Fock space, they give rise to particles and anti-particles of opposite charges.

(g) Expand the fields \( \Phi(x) \) and \( \Phi\dagger(x) \) into the creation and annihilation operators for the particles and antiparticles,

\[ \hat{a}_p = \frac{\hat{a}_{p,1} + i \hat{a}_{p,2}}{\sqrt{2}} \] are particle annihilation operators,

\[ \hat{b}_p = \frac{\hat{a}_{p,1} - i \hat{a}_{p,2}}{\sqrt{2}} \] are antiparticle annihilation operators,

\[ \hat{a}_p^\dagger = \frac{\hat{a}_{p,1} - i \hat{a}_{p,2}}{\sqrt{2}} \] are particle creation operators,

\[ \hat{b}_p^\dagger = \frac{\hat{a}_{p,1} + i \hat{a}_{p,2}}{\sqrt{2}} \] are antiparticle creation operators.

(h) Show that in terms of the operators (12),

\[ \hat{Q}_{21} = -\hat{Q}_{12} = \hat{N}_{\text{particles}} - \hat{N}_{\text{antiparticles}} = \sum_p \left( \hat{a}_p^\dagger \hat{a}_p - \hat{b}_p^\dagger \hat{b}_p \right). \] 

(13)
3. An operator acting on identical bosons can be described in terms of \( N \)-particle wave functions (the \textit{first-quantized} formalism) or in terms of creation and annihilation operators in the Fock space (the \textit{second-quantized} formalism). This problem is about converting the operators from one formalism to another.

Let’s start with a discrete orthonormal basis \( \{ |\alpha\rangle \} \) of single-particle wave states with wave-functions \( \phi_\alpha(x) \). (By abuse of notations, \( x = (x, y, z, \text{spin}, \text{etc.}) \). The corresponding basis of the \( N \)-boson Hilbert space comprises the states

\[
|\alpha, \beta, \cdots, \omega\rangle = \frac{1}{\sqrt{T}} \hat{a}_\omega^\dagger \cdots \hat{a}_\beta^\dagger \hat{a}_\alpha^\dagger |0\rangle
\]  

with totally-symmetrized wave functions

\[
\phi_{\alpha\beta\cdots\omega}(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{D}} \sum_{\text{distinct permutations of } (\alpha, \beta, \ldots, \omega)} \phi_\alpha(x_1) \times \phi_\beta(x_2) \times \cdots \times \phi_\omega(x_N)
\]

\[
= \frac{1}{\sqrt{T \times N!}} \sum_{\text{all } N! \text{ permutations of } (\alpha, \beta, \ldots, \omega)} \phi_\alpha(x_1) \times \phi_\beta(x_2) \times \cdots \times \phi_\omega(x_N),
\]

where \( T = \prod_{\gamma} n_\gamma! \) is the number of trivial permutations between \textit{coincident} entries of the list \( (\alpha, \beta, \ldots, \omega) \) (for example, \( \alpha \leftrightarrow \beta \) when \( \alpha \) and \( \beta \) happen to be equal), and \( D = N!/T \) is the number of \textit{distinct permutations}.

To make sure that the states (14) have the wavefunctions (15), the wave-function picture of the creation and the annihilation operators should be as follows: Given an \( N \)-boson state \( |N, \psi\rangle \) with a totally-symmetric wavefunction \( \psi(x_1, \ldots, x_N) \), the state \( |N + 1, \psi'\rangle = \hat{a}_\alpha^\dagger |N, \psi\rangle \) has a totally-symmetric \( (N + 1) \)-particle wave function

\[
\psi'(x_1, \ldots, x_{N+1}) = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} \phi_\alpha(x_i) \times \psi(x_1, \ldots, \hat{x}_i, \ldots, x_{N+1}).
\]

In particular, for \( N = 0 \), \( \psi'(x_1) = \phi_\alpha(x_1) \). Also,
the state \( |N - 1, \psi''\rangle = \hat{a}_\alpha |N, \psi\rangle \) has a totally-symmetric \( (N - 1) \)-particle wave function

\[
\psi''(x_1, \ldots, x_{N-1}) = \sqrt{N} \int d^3 x_N \phi_\alpha^*(x_N) \times \psi(x_1, \ldots, x_{N-1}, x_N).
\]  

(17)

In particular, for \( N = 1 \), \( \psi'' \) (no arguments) = \( \langle \phi_\alpha | \psi \rangle \). Also, for \( N = 0 \) we simply define \( \hat{a}_\alpha |0\rangle \equiv 0 \).

(a) Verify the commutation relations (1) for these operators.

(b) Verify that the \( \hat{a}_\alpha \) and the \( \hat{a}_\alpha^\dagger \) are hermitian conjugates of each other by checking that

\[
\langle N - 1, \tilde{\psi} | \hat{a}_\alpha | N, \psi \rangle = \langle N, \psi | \hat{a}_\alpha^\dagger | N - 1, \tilde{\psi} \rangle^* 
\]  

(18)

for any \( N \geq 1 \) and any totally-symmetric wave functions \( \psi(x_1, \ldots, x_N) \) and \( \tilde{\psi}(x_1, \ldots, x_{N-1}) \).

(c) Verify that the states (14) indeed have the wavefunctions (15).

* * *

Now let’s move on to the the next subject, namely the one-body operators — the additive operators acting on one particle at a time. In the first-quantized formalism they act on \( N \)-particle states according to

\[
\hat{A}_{\text{net}}^{(1)} = \sum_{i=1}^{N} \hat{A}_1(i^{\text{th}} \text{ particle})
\]  

(19)

where \( \hat{A}_1 \) is some kind of a one-particle operator (such as momentum \( \hat{p} \), or kinetic energy \( \frac{1}{2m} \hat{p}^2 \), or potential \( V(\hat{x}) \), etc., etc.). In the second-quantized formalism such operators become

\[
\hat{A}_{\text{net}}^{(2)} = \sum_{\alpha,\beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta.
\]  

(20)

(d) Verify that the two operators have the same matrix elements between any two \( N \)-boson states \( |N, \psi\rangle \) and \( |N, \tilde{\psi}\rangle \), \( \langle N, \tilde{\psi} | \hat{A}_{\text{net}}^{(1)} | N, \psi \rangle = \langle N, \tilde{\psi} | \hat{A}_{\text{net}}^{(2)} | N, \psi \rangle \).

Hint: use \( \hat{A}_1 = \sum_{\alpha,\beta} \langle \alpha | \hat{A}_1 | \beta \rangle \langle \beta | \).
(e) Now let \( \hat{A}_{\text{net}}^{(2)}, \hat{B}_{\text{net}}^{(2)}, \) and \( \hat{C}_{\text{net}}^{(2)} \) be three second-quantized net one-body operators corresponding to the single-particle operators \( \hat{A}_1, \hat{B}_1, \) and \( \hat{C}_1 \).

Show that if \( \hat{C}_1 = [\hat{A}_1, \hat{B}_1] \) then \( \hat{C}_{\text{net}}^{(2)} = [\hat{A}_{\text{net}}^{(2)}, \hat{B}_{\text{net}}^{(2)}] \).

Finally, consider the two-body operators, i.e., additive operators acting on two particles at a time. Given a two-particle operator \( \hat{B}_2 \) — such as \( V(\hat{x}_1 - \hat{x}_2) \) — the net \( B \) operator acts in the first-quantized formalism according to

\[
\hat{B}_{\text{net}}^{(1)} = \frac{1}{2} \sum_{i \neq j} \hat{B}_2(i^{\text{th}} \text{ and } j^{\text{th}} \text{ particles}),
\]

and in the second-quantized formalism according to

\[
\hat{B}_{\text{net}}^{(2)} = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} (\langle \alpha \mid \otimes \langle \beta \mid) \hat{B}_2 (\mid \gamma \rangle \otimes \mid \delta \rangle) \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta.
\]

Note: in this formula, it is OK to use the un-symmetrized 2-particle states \( \langle \alpha \mid \otimes \langle \beta \mid \) and \( \mid \gamma \rangle \otimes \mid \delta \rangle \), and hence the un-symmetrized matrix elements of the \( \hat{B}_2 \). At the level of the second-quantized operator \( \hat{B}_{\text{net}}^{(2)} \), the Bose symmetry is automatically provided by \( \hat{a}_\alpha^\dagger \hat{a}_\beta = \hat{a}_\beta^\dagger \hat{a}_\gamma \) and \( \hat{a}_\delta \hat{a}_\gamma = \hat{a}_\gamma \hat{a}_\delta \), even for un-symmetrized matrix elements of the \( \hat{B}_2 \).

(f) Similar to part (d), show the operators (21) and (22) have the same matrix elements between any two \( N \)-boson states, \( \langle N, \bar{\psi} \mid \hat{A}_{\text{net}}^{(1)} \mid N, \psi \rangle = \langle N, \bar{\psi} \mid \hat{A}_{\text{net}}^{(2)} \mid N, \psi \rangle \) for any \( \langle N, \bar{\psi} \mid \) and \( \mid N, \psi \rangle \).

(g) Now let \( \hat{A}_1 \) be a one-particle operator, let \( \hat{B}_2 \) and \( \hat{C}_2 \) be two-body operators, and let \( \hat{A}_{\text{net}}^{(2)}, \hat{B}_{\text{net}}^{(2)}, \) and \( \hat{C}_{\text{net}}^{(2)} \) be the corresponding second-quantized operators according to eqs. (20) and (22).

Show that if \( \hat{C}_2 = \left[ (\hat{A}_1(1^{\text{st}}) + \hat{A}_1(2^{\text{nd}})) , \hat{B}_2 \right] \) then \( \hat{C}_{\text{net}}^{(2)} = [\hat{A}_{\text{net}}^{(2)}, \hat{B}_{\text{net}}^{(2)}] \).