This homework is about discrete symmetries of Dirac fermions, the charge conjugation $C$ and the parity (reflection of space) $P$.

1. Let’s start with the charge conjugation $C$ which exchanges particles with antiparticles, for example the electrons $e^-$ with the positrons $e^+$,

$$\hat{C} |e^-(p, s)\rangle = |e^+(p, s)\rangle, \quad \hat{C} |e^+(p, s)\rangle = |e^-(p, s)\rangle.$$  \hspace{1cm} (1)

Note that the operator $\hat{C}$ is unitary and squares to one (repeating the exchange brings us back to the original particles), hence $\hat{C}^\dagger = \hat{C}^{-1} = \hat{C}$.

(a) In the fermionic Fock space, the $\hat{C}$ operator act on multi-particle states by turning each particle into an antiparticle and vice versa according to eqs. (1). Show that this action implies

$$\hat{C} \hat{a}^\dagger_{p,s} \hat{C} = \hat{b}^\dagger_{p,s}, \quad \hat{C} \hat{b}_{p,s} \hat{C} = \hat{a}_{p,s}, \quad \hat{C} \hat{a}_{p,s} \hat{C} = \hat{b}_{p,s}, \quad \hat{C} \hat{b}_{p,s} \hat{C} = \hat{a}_{p,s}.$$  \hspace{1cm} (2)

(b) The quantum Dirac fields $\hat{\Psi}(x)$ and $\hat{\bar{\Psi}}(x)$ are linear combinations of creation and annihilation operators. Use eqs. (2) and the plane-wave relations $v(p, s) = \gamma^2 u^*(p, s)$ and $u(p, s) = \gamma^2 v^*(p, s)$ from the homework set#7 to show that

$$\hat{C} \hat{\Psi}(x) \hat{C} = \gamma^2 \hat{\bar{\Psi}}^*(x) \quad \text{and} \quad \hat{C} \hat{\bar{\Psi}}(x) \hat{C} = \hat{\Psi}^*(x) \gamma^2$$  \hspace{1cm} (3)

where * stands for an hermitian conjugation of the component fields but without transposing a column vector (of 4 Dirac components) into a row vector or vice versa, thus

$$\hat{\Psi} = \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \hat{\psi}_3 \\ \hat{\psi}_4 \end{pmatrix}, \quad \hat{\Psi}^* = \begin{pmatrix} \hat{\psi}_1^\dagger \\ \hat{\psi}_2^\dagger \\ \hat{\psi}_3^\dagger \\ \hat{\psi}_4^\dagger \end{pmatrix}, \quad \hat{\bar{\Psi}} = \begin{pmatrix} \hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4 \end{pmatrix} \times \gamma^0,$$  \hspace{1cm} (4)

$$\hat{\bar{\Psi}}^* = \begin{pmatrix} \hat{\psi}_1^\dagger, \hat{\psi}_2^\dagger, \hat{\psi}_3^\dagger, \hat{\psi}_4^\dagger \end{pmatrix} \times \gamma^0.$$

(c) Show that the Dirac equation transforms covariantly under the charge conjugation (3). Hint: prove and use $\gamma^\mu \gamma^2 = -\gamma^2 (\gamma^\mu)^*$ for all $\gamma^\mu$ in the Weyl basis.
(d) Show that the classical Dirac Lagrangian is invariant under the charge conjugation (up to a total spacetime derivative). Note that in the classical limit the Dirac fields anticommute with each other, $\Psi_\alpha^\dagger \Psi_\beta = -\Psi_\beta^\dagger \Psi_\alpha$. Also, similar to the hermitian conjugation of quantum fields, the complex conjugation of fermionic fields reverses their order: $(F_1F_2)^* = F_2^*F_1^* = -F_1^*F_2^*$.

2. Now consider the parity $P$, the improper Lorentz symmetry that reflects the space but not the time, $(x, t) \rightarrow (-x, +t)$. This symmetry acts on Dirac spinor fields according to

$$\hat{\Psi}'(-x, +t) = \pm \gamma^0 \hat{\Psi}(+x, +t) \quad (5)$$

where the overall $\pm$ sign is intrinsic parity of the fermion species.

(a) Verify that the Dirac equation transforms covariantly under (5) and that the Dirac Lagrangian is invariant (apart from $\mathcal{L}(x, t) \rightarrow \mathcal{L}(-x, t)$).

In the Fock space, eq. (5) becomes

$$\hat{P} \hat{\Psi}(x, t) \hat{P} = \pm \gamma^0 \hat{\Psi}(-x, t) \quad (6)$$

for some unitary operator $\hat{P}$ that squares to one. Let’s find how this operator acts on the particles and their states.

(b) First, look up the plane-wave solutions $u(p, s)$ and $v(p, s)$ in the homework set #7 and show that $u(-p, s) = +\gamma^0 u(p, s)$ while $v(-p, s) = -\gamma^0 v(p, s)$.

(c) Now show that eq. (6) implies

$$\hat{P} \hat{a}_{p, s} \hat{P} = \pm \hat{a}_{-p, +s}, \quad \hat{P} \hat{a}_{p, s}^\dagger \hat{P} = \pm \hat{a}_{-p, +s}^\dagger,$$

$$\hat{P} \hat{b}_{p, s} \hat{P} = \mp \hat{b}_{-p, +s}, \quad \hat{P} \hat{b}_{p, s}^\dagger \hat{P} = \mp \hat{b}_{-p, +s}^\dagger \quad (7)$$

and hence

$$\hat{P} |F(p, s)\rangle = \pm |F(-p, +s)\rangle \quad \text{and} \quad \hat{P} |\overline{F}(p, s)\rangle = \mp |\overline{F}(-p, +s)\rangle. \quad (8)$$

Note that the fermion $F$ and the antifermion $\overline{F}$ have opposite intrinsic parities.
3. Some electrically neutral particles carry other kinds of changes (forex, the baryon number) that distinguish them from their antiparticles. But other particles — such as the photon or the $\pi^0$ meson — have no charges at all and act as their own antiparticles. The charge conjugation symmetry turns such particles $n$ into themselves,

$$\hat{C}|n(p, s)\rangle = \pm |n(p, s)\rangle,$$

(9)

where the overall $\pm$ sign is called the $C$-parity or charge-parity of the particle in question. This $C$-parity — as well as the $P$-parity under space reflections — limit the allowed decay channels of unstable particles via strong and EM interactions which respect both $\hat{C}$ and $\hat{P}$ symmetries.

Consider a bound state of a charged Dirac fermion $F$ and the corresponding antifermion, for example a $q\bar{q}$ meson or a positronium “atom” (a hydrogen-atom-like bound state of $e^−$ and $e^+\bar{}$). In the Fock space of fermions and antifermions, such bound state with zero net momentum obtains as

$$|B(p_{tot} = 0)\rangle = \int \frac{d^3p_{red}}{(2\pi)^3} \sum_{s_1, s_2} \psi(p_{red}, s_1, s_2) \times \hat{a}^\dagger(+p_{red}, s_1) \hat{b}^\dagger(-p_{red}, s_2) |0\rangle$$

(10)

for some wave-function $\psi$ of the reduced momentum and the two spins.

Suppose this bound state has a definite orbital angular momentum $L$ — which controls the symmetry of the wave function $\psi$ with respect to $p_{red} \rightarrow -p_{red}$ — and definite net spin $S$ — which controls the symmetry of $\psi$ under $s_1 \leftrightarrow s_2$. Turns out that the $L$ and the $S$ of the bound state also determine its $C$-parity and $P$-parity.

(a) Show that $C = (-1)^{L+S}$.

(b) Show that $P = (-1)^{L+1}$.

Now let’s apply these results to the positronium — a hydrogen-atom-like bound state of a positron $e^+$ and an electron $e^-$. The ground state of positronium is hydrogen-like $1S$ ($n = 1, L = 0$), with the net spin which could be either $S = 0$ or $S = 0$. 

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(c) Explain why the $S = 0$ state annihilates into photons much faster than the $S = 1$ state.

Hint#1: Annihilation rate of positronium into $n$ photons happens in the $n^{\text{th}}$ order of QED perturbation theory, so the rate $\propto \alpha^n$ (for $\alpha \approx 1/137$).

Hint#2: Since the EM fields couple linearly to the electric charges and currents (which are reversed by $\hat{C}$), each photon has $C = -1$.

4. A Dirac spinor field $\Psi(x)$ comprises two 2-component Weyl spinor fields,

$$\hat{\Psi}(x) = \begin{pmatrix} \hat{\psi}_L(x) \\ \hat{\psi}_R(x) \end{pmatrix}. \quad (11)$$

Spell out the actions of the C, P, and the combined CP symmetry on the Weyl spinors. In particular, show that C and P interchange the two spinors, while the combined CP symmetry acts on the $\psi_L$ and the $\psi_R$ independently from each other.

5. Finally, consider bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi}\Gamma\Psi$ where $\Gamma$ is one of 16 matrices discussed in the previous homework; altogether, we have

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}\frac{i}{2}\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi, \quad \text{and} \quad P = \overline{\Psi}i\gamma^5\Psi. \quad (12)$$

(a) Show that all the bilinears (12) are Hermitian.

Hint: First, show that $(\overline{\Psi}\Gamma\Psi)\dagger = \overline{\Psi}\Gamma\Psi$. 

Note: despite the Fermi statistics, $(\Psi_\alpha^\dagger\Psi_{\beta})\dagger = +\Psi_\beta^\dagger\Psi_{\alpha}$.

(b) Show that under continuous Lorentz symmetries, the $S$ and the $P$ transform as scalars, the $V^\mu$ and the $A^\mu$ as vectors, and the $T^{\mu\nu}$ as an antisymmetric tensor.

(c) Find the transformation rules of the bilinears (12) under parity and show that while $S$ is a true scalar and $V$ is a true (polar) vector, $P$ is a pseudoscalar and $A$ is an axial vector.
Next, consider the charge-conjugation properties of the Dirac bilinears. To avoid the operator-ordering problems, take the classical limit where $\Psi(x)$ and $\Psi^\dagger(x)$ *anticommut*e with each other, $\Psi_\alpha \Psi^\dagger_\beta = -\Psi^\dagger_\beta \Psi_\alpha$.

(d) Show that $C$ turns $\bar{\Psi} \Gamma \Psi$ into $\bar{\Psi} \Gamma^c \Psi$ where $\Gamma^c = \gamma^0 \gamma^2 \Gamma^\top \gamma^0 \gamma^2$.

(e) Calculate $\Gamma^c$ for all 16 independent matrices $\Gamma$ and find out which Dirac bilinears are $C$–even and which are $C$–odd.