Problem 1(a):
Let $\Delta T^{\mu \nu} = \partial_\lambda K^{\lambda \mu \nu}$. Regardless of the specific form of the $K^{\lambda \mu \nu}(\phi, \partial \phi)$ tensor, its anti-symmetry with respect to its first two indices $K^{\lambda \mu \nu} \equiv -K^{\mu \lambda \nu}$ implies

$$\partial_\mu \Delta T^{\mu \nu} = \partial_\mu \partial_\lambda K^{\lambda \mu \nu} = 0$$  \hspace{1cm} (S.1)

and hence

$$\partial_\mu T^{\mu \nu} = \partial_\mu T^{\mu \nu}_{\text{Noether}} = (\text{hopefully}) = 0. \hspace{1cm} (3.a)$$

Furthermore,

$$\int d^3x \left( \Delta T^{0 \nu} = \partial_i K^{i0 \nu} \right) = \oint \text{boundary of space} \ d^2 \text{Area} \ iK^{i0 \nu} \rightarrow 0 \hspace{1cm} (S.2)$$

when the integral is taken over the whole space, hence

$$P_{\text{net}}^{\nu} \equiv \int d^3x T^{0 \nu} = \int d^3x t^{0 \nu}_{\text{Noether}}. \hspace{1cm} (3.b)$$

Problem 1(b):
In the Noether’s formula (1) for the stress-energy tensor, $\phi_a$ stand for the independent fields, however labeled. In the electromagnetic case, the independent fields are components of the 4-vector $A_\lambda(x)$, hence

$$T^{\mu \nu}_{\text{Noether}}(\text{EM}) = \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu \nu} \mathcal{L}$$

$$= -F^{\mu \lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu \nu} F_{\kappa \lambda} F^{\kappa \lambda}. \hspace{1cm} (S.3)$$

While the second term here is clearly both gauge invariant and symmetric in $\mu \leftrightarrow \nu$, the first term is neither.
Problem 1(c):
Clearly, one can easily restore both symmetry and gauge invariance of the electromagnetic stress-energy tensor by replacing \( \partial^\nu A_\lambda \) in eq. (S.3) with \( F^\nu_\lambda \), hence eq. (5). The correction to the \( T_{\text{Noether}}^{\mu\nu} \) amounts to

\[
\Delta T^{\mu\nu} = T^{\mu\nu}_{\text{phys}} - T^{\mu\nu}_{\text{Noether}} = -F^{\mu\lambda} F^\nu_\lambda + F^{\nu\lambda} \partial^\nu A_\lambda \\
= -F^{\mu\lambda} \times (F^\nu_\lambda - \partial^\nu A_\lambda) = +F^{\mu\lambda} \times \partial_\lambda A^\nu \\
= \partial_\lambda \left( F^{\mu\lambda} A^\nu \right) - A^\nu \times \partial_\lambda F^{\mu\lambda}.
\]

(S.4)

On the bottom line here, the second term vanishes for the free EM field, which satisfies \( \partial_\lambda F^{\mu\lambda} = 0 \). The remaining first term has the form of a total derivative of a 3-index tensor \( F^{\mu\lambda} A^\nu \) which is antisymmetric in the first two indices; in another words, it has the requisite form of \( \partial_\lambda K^{\lambda\mu\nu} \). Altogether, we have

\[
T^{\mu\nu}_{\text{phys}} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda K^{\lambda\mu\nu} \quad \text{where} \quad K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu = -K^{\mu\lambda\nu}
\]

(S.5)
in perfect agreement with eq. (2).

Problem 1(d):
Let’s start with the Lagrangian (4). In component form,

\[
F^{i0} = -F^{0i} = E^i, \quad F^{ij} = -\epsilon^{ijk} B^k.
\]

(S.6)

Therefore, \( F^{i0} F_{i0} = F^{0i} F_{0i} = -E^i E^i \) where the minus sign comes from raising one space index. Likewise, \( F^{ij} F_{ij} = +\epsilon^{ijk} B^k \epsilon^{ij\ell} B^\ell = +2B^k B^k \) where the plus sign comes from raising two space indices at once. Thus,

\[
\mathcal{L} = -\frac{1}{4} \left( F^{\mu\nu} F_{\mu\nu} = F^{i0} F_{i0} + F^{0i} F_{0i} + F^{ij} F_{ij} \right) = \frac{1}{2} (E^2 - B^2).
\]

(S.7)

Now let’s work out the stress-energy tensor (4) in components. For the energy density component \( T^{00} \), we have

\[
T^{00} = -F^{0i} F^0_i - \mathcal{L} = +E^2 - \frac{1}{2} (E^2 - B^2) = \frac{1}{2} (E^2 + B^2)
\]

(S.8)
in agreement with the standard electromagnetic formulæ (in the rationalized \( c = 1 \) units).
Likewise, the energy flux and the momentum density are

\[
S^i \equiv T^{0i} = T^{0i} = -F^{0j}F^i_j = -(E^j)(+\varepsilon^{ijk}B^k) = +\varepsilon^{ijk}E^jB^k = (E \times B)^i, \quad (S.9)
\]

in agreement with the Poynting vector \( \mathbf{S} = \mathbf{E} \times \mathbf{B} \) (again, in the rationalized \( c = 1 \) units).

Finally, the (3-dimensional) stress tensor is

\[
T^{ij}_{EM} = -F^{i\lambda}F^j_\lambda - g^{ij}\mathcal{L} = -F^{0i}F^j_0 - F^{ik}F^j_k + \delta^{ij}\mathcal{L}
\]

\[
= -E^iE^j + \varepsilon^{ik\ell}B^\ell e^{jkm}B^m + \frac{1}{2}\delta^{ij}(E^2 - B^2)
\]

\[
= -E^iE^j + (\delta^{ij}B^\ell B^\ell - B^iB^j) + \frac{1}{2}\delta^{ij}(E^2 - B^2)
\]

\[
= -E^iE^j - B^iB^j + \frac{1}{2}\delta^{ij}(E^2 + B^2).
\]

(S.10)

Note the anisotropy of this stress tensor.

**Problem 1(e):**

In a sense, eq. (7) follows from eq. (S.4), but it is just as easy to derive it directly from Maxwell equations. Starting with eq. (4), we immediately have

\[
\partial_\mu T^{\mu\nu}_{EM} = -(\partial_\mu F^{\mu\lambda})F^\nu_\lambda - F^{\mu\lambda}(\partial_\mu F^\nu_\lambda) + \frac{1}{2}F_{\kappa\lambda}(\partial^\nu F^{\kappa\lambda}).
\]

(S.11)

Let’s simplify the second term here:

\[
-F^{\mu\lambda} \times \partial_\mu F^\nu_\lambda = -F^{\mu\lambda} \times \partial^\mu F^{\nu_\lambda}
\]

\[
\langle\text{exchanging summation indices } \mu \leftrightarrow \lambda \rangle
\]

\[
= -F_{\lambda\mu} \times \partial^\lambda F^{\nu\mu}
\]

\[
= +F_{\mu\lambda} \times \partial^\lambda F^{\nu\mu}
\]

(S.12)

\[
= +\frac{1}{2}F_{\mu\lambda} \times \partial^\lambda F^{\nu\mu} - \frac{1}{2}F_{\mu\lambda} \times \partial^\mu F^{\nu\lambda}
\]

\[
= +\frac{1}{2}F_{\mu\lambda} \times (\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu})
\]

\[
= -\frac{1}{2}F_{\mu\lambda} \times \partial^\nu F^{\mu\lambda},
\]

where the last line here follows from the homogeneous Maxwell equation

\[
\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} = 0.
\]

(S.13)

Thanks to eq. (S.12), the second term in eq. (S.11) cancels the third term, and we are left
with the first term only. Thus,

\[ \partial_\mu T_{EM}^{\mu\nu} = - (\partial_\mu F^{\mu\lambda}) F^\nu_\lambda = - J^\lambda F^\nu_\lambda \]  

(S.14)

where the second equality comes from the in-homogeneous Maxwell equation \( \partial_\mu F^{\mu\lambda} = J^\lambda \).

This proves eq. (7). Finally, eq. (8) follows from eq. (7) and the net stress-energy conservation (6).  \( \text{Q.E.D.} \)

Problem 2(a):
As discussed in class, Euler–Lagrange equations for charged fields can be written in a manifestly covariant form as

\[ D_\mu \frac{\partial L}{\partial (D_\mu \phi)} - \frac{\partial L}{\partial \phi} = 0. \]  

(S.15)

In particularly, for \( \phi = \Phi \), we have

\[ \frac{\partial L}{\partial (D_\mu \Phi)} = D^\mu \Phi^*, \quad \frac{\partial L}{\partial \Phi} = -m^2 \Phi^*, \]

which gives us

\[ D_\mu D^\mu \Phi^* + m^2 \Phi^* = 0. \]  

(S.16)

Likewise, for \( \phi = \Phi^* \) we have

\[ \frac{\partial L}{\partial (D_\mu \Phi^*)} = D^\mu \Phi, \quad \frac{\partial L}{\partial \Phi^*} = -m^2 \Phi, \]

and therefore

\[ D_\mu D^\mu \Phi + m^2 \Phi = 0. \]  

(S.17)

As for the vector fields \( A_\nu \), the Lagrangian (9) depends on \( \partial_\mu A_\nu \) only through \( F^{\mu\nu} \), which gives us the usual Maxwell equation

\[ \partial_\mu F^{\mu\nu} = J^\nu \quad \text{where} \quad J^\nu \equiv - \frac{\partial L}{\partial A_\nu}. \]  

(S.18)

To obtain the current \( J^\nu \), we notice that the covariant derivatives of the charged fields \( \Phi \)
and $\Phi^*$ depend on the gauge field:

$$\frac{\partial D_{\mu} \Phi}{\partial A_\nu} = iq^\nu \Phi, \quad \frac{\partial D_{\mu} \Phi^*}{\partial A_\nu} = -iq^\nu \Phi^*. \quad (S.19)$$

Consequently,

$$J^\nu = -\frac{\partial L}{\partial D_{\nu} \Phi} \times (iq \Phi) - \frac{\partial L}{\partial D_{\nu} \Phi^*} \times (-iq \Phi^*) $$

$$= -iq \Phi \times D^\nu \Phi^* + iq \Phi^* \times D^\nu \Phi. \quad (S.20)$$

Note that all derivatives on the last line here are gauge-covariant, which makes the current $J^\nu$ gauge invariant. In a non-covariant form,

$$J^\nu = iq \Phi^* D^\nu \Phi - iq \Phi D^\nu \Phi^* - 2q^2 \Phi^* \Phi A^\nu. \quad (S.21)$$

To prove the conservation of this current, we use the Leibniz rule for the covariant derivatives, $D_\nu (XY) = XD_\nu Y + YD_\nu X$. This gives us

$$\partial_\mu (\Phi^* D^\mu \Phi) = D_\mu (\Phi^* D^\mu \Phi) = (D_\mu \Phi^*)(D^\mu \Phi) + \Phi^* (D_\mu D^\mu \Phi),$$

$$\partial_\mu (\Phi D^\mu \Phi^*) = D_\mu (\Phi D^\mu \Phi^*) = (D_\mu \Phi)(D^\mu \Phi^*) + \Phi (D_\mu D^\mu \Phi^*), \quad (S.22)$$

and hence in light of eq. (S.20) for the current,

$$\partial_\nu J^\nu = -iq \left( (D_\nu \Phi)(D^\nu \Phi^*) + \Phi (D_\nu D^\nu \Phi^*) \right) + iq \left( (D_\nu \Phi^*)(D^\nu \Phi) + \Phi^* (D_\nu D^\nu \Phi) \right)$$

$$= iq \Phi^* D^2 \Phi - iq \Phi D^2 \Phi^*$$

(by equations of motion)

$$= iq \Phi^* (\Phi^2) - iq \Phi (\Phi^2)$$

$$= 0. \quad (S.23)$$

**Q.E.D.**
Problem 2(b):
According to the Noether theorem,

\[ T^{\mu\nu}_{\text{Noether}} = \frac{\partial L}{\partial (\partial_\mu A_\lambda)} \partial^\nu A_\lambda + \frac{\partial L}{\partial (\partial_\mu \Phi)} \partial^\nu \Phi + \frac{\partial L}{\partial (\partial_\mu \Phi^*)} \partial^\nu \Phi^* - g^{\mu\nu} L \]  

(S.24)

\[ = T^{\mu\nu}_{\text{Noether}}(\text{EM}) + T^{\mu\nu}_{\text{Noether}}(\text{matter}) \]

where

\[ T^{\mu\nu}_{\text{Noether}}(\text{EM}) = -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \]  

(S.25)

similar to the free EM fields, and

\[ T^{\mu\nu}_{\text{Noether}}(\text{matter}) = D^\mu \Phi^* \partial^\nu \Phi + D^\mu \Phi \partial^\nu \Phi^* - g^{\mu\nu} (D^\lambda \Phi^* D_\lambda \Phi - m^2 \Phi^* \Phi). \]  

(S.26)

Both terms on the second line of eq. (S.24) lack \( \mu \leftrightarrow \nu \) symmetry and gauge invariance and thus need \( \partial_\lambda \mathcal{K}^{\lambda\mu\nu} \) corrections for some \( \mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu} \). We would like to show that the same \( \mathcal{K}^{\lambda\mu\nu} = -F^{\mu\lambda} A^\nu \) we used to improve the free electromagnetic stress-energy tensor will now improve both the \( T^{\mu\nu}_{\text{EM}} \) and \( T^{\mu\nu}_{\text{mat}} \) at the same time!

Indeed, to improve the scalar fields’ stress-energy tensor we need

\[ \Delta T^{\mu\nu}_{\text{matter}} = T^{\mu\nu}_{\text{phys}}(\text{matter}) - T^{\mu\nu}_{\text{Noether}}(\text{matter}) \]

\[ = D^\mu \Phi^* (D^\nu \Phi - \partial^\nu \Phi) + D^\mu \Phi (D^\nu \Phi^* - \partial^\nu \Phi^*) \]

\[ = D^\mu \Phi^* (iq A^\nu \Phi) + D^\mu \Phi (-iq A^\nu \Phi^*) \]  

(S.27)

\[ = -A^\nu \times \left( iq \Phi^* D^\mu \Phi - iq \Phi D^\mu \Phi^* \right) \]

\[ = -A^\nu J^\mu, \]

while the improvement of the EM stress-energy was worked out in problem 1.(c):

\[ \Delta T^{\mu\nu}_{\text{EM}} = -F^{\mu\lambda} (F^\nu_\lambda - \partial^\nu A_\lambda) = +F^{\mu\lambda} \partial_\lambda A^\nu = \partial_\lambda (-F^{\lambda\mu} A^\nu) + A^\nu J^\mu, \]  

(S.28)

cf. eq. (S.4). Note that the \( A^\nu J^\mu \) term cancels between the EM and the matter improvement terms, so the net improvement needed to symmetrize the combined stress-energy tensor is
simply
\[
\Delta T_{\text{tot}}^{\mu \nu} \equiv T_{\text{phys}}^{\mu \nu}(\text{total}) - T_{\text{Noether}}^{\mu \nu}(\text{total}) \\
= \Delta T_{\text{matter}}^{\mu \nu} + \Delta T_{\text{EM}}^{\mu \nu} \\
= \partial_\lambda \left( -F^{\lambda \mu} A^\nu \equiv \mathcal{K}^{\lambda \mu} \right).
\] (S.29)

\text{Q.E.D.}

Problem 2(c):
Because the fields \(\Phi(x)\) and \(\Phi^*(x)\) have opposite electric charges, their product is neutral and therefore
\[
\partial_\mu (\Phi^* \Phi) = D_\mu (\Phi^* \Phi) = (D_\mu \Phi^*) \Phi + \Phi^* (D_\mu \Phi).
\]
Similarly,
\[
\partial_\mu ((D^\mu \Phi^*) (D^\nu \Phi)) = (D_\mu D^\mu \Phi^*) (D^\nu \Phi) + (D^\mu \Phi^*) (D_\mu D^\nu \Phi) \\
= -m^2 \Phi^* (D^\nu \Phi) + (D_\mu \Phi^*) (D^\nu D^\mu \Phi + iqF^{\mu \nu} \Phi)
\] (S.30)
where we have applied the field equation \((D_\mu D^\mu + m^2) \Phi^*(x) = 0\) to the first term on the right hand side and used \([D^\mu, D^\nu] \Phi = iqF^{\mu \nu} \Phi\) to expand the second term. Likewise,
\[
\partial_\mu ((D^\mu \Phi) (D^\nu \Phi^*)) = (D_\mu D^\mu \Phi) (D^\nu \Phi^*) + (D^\mu \Phi) (D_\mu D^\nu \Phi^*) \\
= -m^2 \Phi (D^\nu \Phi^*) + (D_\mu \Phi) (D^\nu D^\mu \Phi^* - iqF^{\mu \nu} \Phi^*)
\] (S.31)
and
\[
\partial_\mu \left[ -g^{\mu \nu} \left( D_\lambda \Phi^* D^\lambda \Phi - m^2 \Phi^* \Phi \right) \right] = -\partial^\nu \left( D_\lambda \Phi^* D^\lambda \Phi \right) + m^2 \partial^\nu (\Phi^* \Phi) \\
= -(D^\nu D^\mu \Phi^*) (D_\mu \Phi) - (D_\mu \Phi^*) (D^\nu D^\mu \Phi) \\
+ m^2 \Phi (D^\nu \Phi^*) + m^2 \Phi^* (D^\nu \Phi).
\] (S.32)
Together, the left hand sides of eqs. (S.30), (S.31) and (S.32) comprise \(\partial_\mu T_{\text{mat}}^{\mu \nu} \) — cf. eq. (27).
On the other hand, combining the right hand sides of these three equations results in massive cancellation of all terms except those containing the gauge field strength tensor \(F^{\mu \nu}\).
Therefore,
\[
\partial_\mu T_{\text{mat}}^{\mu \nu} = (D_\mu \Phi^*) (iqF^{\mu \nu} \Phi) + (D_\mu \Phi) (-iqF^{\mu \nu} \Phi^*) \\
= F^{\mu \nu} (iq \Phi D_\mu \Phi^* - iq \Phi^* D_\mu \Phi) \\
= F^{\mu \nu} \times (-J_\mu) = +F^{\nu \lambda} J_\lambda
\] (S.33)
in accordance with eq. (17).
Finally, combining this formula with eq. (7) we see that it follows that the total stress-energy tensor (12) is conserved,

$$\partial_\mu T^{\mu\nu}_{\text{tot}} = \partial_\mu T^{\mu\nu}_{\text{tot}} + \partial_\mu T^{\mu\nu}_{\text{EM}} = 0.$$  \hfill (S.34)

\textbf{Q.E.D.}

\textbf{Problem 3(a):}

Unitary transform (17) of the $\Phi_a$ fields into each other leaves the bilinear combination $\sum_a \Phi_a^* \Phi_a$ invariant — which immediately leads to the invariance of the scalar potential in the Lagrangian (16). Indeed, a unitary matrix $U$ satisfies $U^\dagger U = 1$ — i.e., $\sum_a (U^\dagger)^b_a \times U^a_c = \delta^a_c$ — and hence

$$\sum_a \Phi_a^* \Phi_a = \sum_a \left( \sum_b \Phi_b^* (U^\dagger)^b_a \right) \left( \sum_c U^a_c \Phi_c \right)$$

$$= \sum_b \Phi_b^* \Phi_b \times \left( \sum_a (U^\dagger)^b_a \times U^a_c = \delta^a_c \right) = \sum_b \Phi_b^* \Phi_b.$$ \hfill (S.35)

Similarly, the kinetic part of the Lagrangian would be invariant \textit{provided the derivatives $D_\mu \Phi_a$ and $D_\mu \Phi_a^*$ transform into each other just like the fields itself},

$$D_\mu \Phi_a \to \sum_b U^a_b \times D_\mu \Phi^b \quad \& \quad D_\mu \Phi_a^* \to \sum_b D_\mu \Phi_b^* \times (U^\dagger)^b_a \quad \implies \quad \sum_a D_\mu \Phi_a^* \times D^\mu \Phi_a \text{ invariant.}$$ \hfill (S.36)

Thus, all we need to prove is the assumptions on these line — or equivalently, that \textit{the covariant derivatives $D_\mu$ commute with the unitary transform (17)}.

For a \textit{global symmetry}, the $U$ matrix is $x$-independent, thus $\partial_\mu U^a_b = 0$ which makes the symmetry transform commute with the ordinary derivatives $\partial_\mu$. To commute with the covariant derivatives

$$D_\mu = \partial_\mu + iA_\mu(x)\hat{Q},$$ \hfill (S.37)

the symmetry transform should also commute with the electric charge operator $Q$. In other words, it should not mix fields having different electric charges.
For the problem at hand, all the $\Phi^a$ fields have the same charge $+q$ while all the conjugate fields $\Phi^*_{a}$ have the opposite charge $-q$. Thus, a symmetry may mix a $\Phi^a$ field with any other $\Phi^b$ fields — as in eq. (17) — but not with any of the $\Phi^*_{b}$ conjugate fields. Likewise, it may mix a $\Phi^*_{a}$ with the other $\Phi^*_{b}$ but now with any of the $\Phi^b$.

**Problem 3(b):**

Consider an infinitesimal unitary symmetry $U = 1 + i\epsilon T + O(\epsilon^2)$ for some hermitian matrix $T = T^\dagger$. The generator $T$ of this symmetry acts on the fields as

$$
\delta \Phi^a = \epsilon T \Phi^a = \epsilon \sum_b iT^a_b \Phi^b, \quad \delta \Phi^*_{a} = \epsilon T \Phi^*_{a} = \epsilon \sum_b (-i) \Phi^*_{b} T^b_a,
$$

so the Noether current of this symmetry is

$$
J_T^\mu = \sum_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^a)} \times T \Phi^a + \sum_a \frac{\partial \mathcal{L}}{\partial (\partial^*_\mu \Phi^*_{a})} \times T \Phi^*_{a}
$$

$$
= \sum_a D^\mu \Phi^*_{a} \times \sum_b iT^a_b \Phi^b + \sum_a D^\mu \Phi^a \times \sum_b (-i) \Phi^*_{b} T^b_a
$$

$$
= \sum_{ab} T^a_b \times \left( i \Phi^b \times D^\mu \Phi^*_{a} - i \Phi^*_{a} \times D^\mu \Phi^a \right)
$$

$$
= \sum_{ba} T^a_b \times (J^\mu)^b_a
$$

$$
= \text{tr} \left(TJ^\mu\right).
$$

where $J^\mu$ is the hermitian matrix of currents

$$
(J^\mu)^b_a = i \Phi^b D^\mu \Phi^*_{a} - i \Phi^*_{a} D^\mu \Phi^b = (J^\mu)^*_{a} b.
$$

Note: a different hermitian matrix $T'$ would generate a different infinitesimal unitary symmetry $U' = 1 + i\epsilon T'$, which in turn would have a different Noether current $J'_{T'} = \text{tr} \left(T'J^\mu\right)$. However, all such currents are linear combinations of the same $N^2$ currents $(J^\mu)^a_b$, and there is obvious linear between a particular hermitian generator $T$ and the corresponding Noether current $J_T$. Consequently, the whole matrix $J^\mu = \| (J^\mu)^a_b \|$ acts as a matrix of Noether currents of the $U(N)$ symmetry.
Problem 3(c):
Since the product \( \Phi^a \times D^\mu \Phi^*_b \) is electrically neutral, we have

\[
\partial_\mu \left( \Phi^a D^\mu \Phi^*_b \right) = D_\mu \left( \Phi^a D^\mu \Phi^*_b \right) = (D_\mu \Phi^a) \left( D^\mu \Phi^*_b \right) + \Phi^a \left( D_\mu D^\mu \Phi^*_b \right),
\]

(S.40)

and likewise

\[
\partial_\mu \left( \Phi^*_b D^\mu \Phi^a \right) = D_\mu \left( \Phi^*_b D^\mu \Phi^a \right) = (D_\mu \Phi^*_b) \left( D^\mu \Phi^a \right) + \Phi^*_b \left( D_\mu D^\mu \Phi^a \right).
\]

(S.41)

Consequently

\[
\partial_\mu \left( \Phi^a D^\mu \Phi^*_b - \Phi^*_b D^\mu \Phi^a \right) = \Phi^a \left( D_\mu D^\mu \Phi^*_b \right) - \Phi^*_b \left( D_\mu D^\mu \Phi^a \right)
\]

(S.42)

while the remaining terms in eqs. (S.40) and (S.41) cancel each other, which means

\[
\partial_\mu J^{\mu a}_{\ b} = i\Phi^a \left( D_\mu D^\mu \Phi^*_b \right) - i\Phi^*_b \left( D_\mu D^\mu \Phi^a \right).
\]

(S.43)

Moreover, when the scalar fields obey their equations of motion

\[
D_\mu D^\mu \Phi^*_a = -\frac{\partial V}{\partial \Phi^a} = -\Phi^*_a \times \left( m^2 + \frac{\lambda}{2} \sum_{b} |\Phi^b|^2 \right),
\]

\[
D_\mu D^\mu \Phi^a = -\frac{\partial V}{\partial \Phi^*_a} = -\Phi^a \times \left( m^2 + \frac{\lambda}{2} \sum_{b} |\Phi^b|^2 \right),
\]

(S.44)

the right hand sides of eqs. (S.43) vanish altogether and the Noether currents (18) are conserved:

\[
\partial_\mu J^{\mu a}_{\ b} = i\Phi^a \left( D_\mu D^\mu \Phi^*_b \right) - i\Phi^*_b \left( D_\mu D^\mu \Phi^a \right)
\]

\[
= i\Phi^a \times \Phi^*_b \times \left( m^2 + \frac{\lambda}{2} \sum_{b} |\Phi^b|^2 \right) - i\Phi^*_b \times \Phi^a \times \left( m^2 + \frac{\lambda}{2} \sum_{b} |\Phi^b|^2 \right)
\]

(S.45)

\[
= 0.
\]

Q.E.D.
Problem 3(d):

$N$ complex fields $\Phi_a(x)$ are equivalent to $2N$ real fields $\phi_{a,\alpha}(x)$ for $a = 1, \ldots, N$ and $\alpha = 1, 2$:

$$\Phi^a(x) = \frac{\phi_{a1}(x) + i\phi_{a2}(x)}{\sqrt{2}}, \quad \Phi^*_a(x) = \frac{\phi_{a1}(x) - i\phi_{a2}(x)}{\sqrt{2}}.$$ (S.46)

The scalar potential part of the Lagrangian (16) is a function of

$$\sum_a \Phi^*_a \Phi^a = \sum_a \frac{\phi_{a1}^2 + \phi_{a2}^2}{2} = \frac{1}{2} \sum_{a,\alpha} \phi_{a,\alpha}^2$$ (S.47)

so it is invariant under any $SO(2N)$ rotation of the $2N$ real fields. For electrically neutral fields, the net kinetic part of the Lagrangian

$$\mathcal{L}^{\text{kinetic}}_{\text{neutral}} = \sum_a \partial_{\mu} \Phi^*_a \partial^{\mu} \Phi^a = \sum_{a,\alpha} \frac{1}{2} (\partial_{\mu} \phi_{a,\alpha})^2$$ (S.48)

would also be invariant under any $SO(2N)$ symmetry (as long as the transformations are global, i.e., $x$–independent), but for the electrically charged fields the situation is more complicated.

Indeed, for a charged field $\Phi$, the gauge-covariant kinetic term $D_\mu \Phi^* D^{\mu} \Phi$ comprises both the kinetic terms per se but also the coupling of the charged field to the EM fields $A^\mu$:

$$D_\mu \Phi^* D^{\mu} \Phi = (\partial_\mu \Phi^* - iqA_\mu \Phi^*) (\partial^{\mu} \Phi + iqA_\mu)$$

$$= \partial_\mu \Phi^* \partial^{\mu} \Phi + iqA_\mu \times (\Phi \partial^{\mu} \Phi^* - \Phi^* \partial^{\mu} \Phi) + q^2 A_\mu A^\mu \times \Phi^* \Phi.$$ (S.49)

In terms of the two real fields $\phi_\alpha$ comprising the complex field $\Phi$, this formula becomes

$$D_\mu \Phi^* D^{\mu} \Phi = \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + qA_\mu \times (\phi_2 \partial^{\mu} \phi_1 - \phi_1 \partial^{\mu} \phi_2) + \frac{1}{2} q^2 A_\mu A^\mu \times (\phi_1^2 + \phi_2^2).$$ (S.50)

Consequently, for the $2N$ real fields comprising the $N$ complex fields of similar electric charge...
we have

\[ L_{\text{kinetic charged}} = \sum_a D_\mu \Phi_\alpha^* D^\mu \Phi^a \]
\[ = \frac{1}{2} \sum_{a,a} (\partial_\mu \phi_{a,\alpha})^2 - q A_\mu \times \sum_a \sum_{\alpha,\beta} E_{\alpha\beta} \phi_{a,\alpha} \partial_\mu \phi_{a,\beta} + \frac{1}{2} q^2 A_\mu A^\mu \times \sum_{a,\alpha} \phi_{a,\alpha}^2 \]

where \( E_{\alpha\beta} \) is a 2 \( \times \) 2 antisymmetric matrix

\[ E = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \quad \text{that is } E_{11} = E_{22} = 0, \ E_{12} = +1, \ E_{21} = -1. \quad (S.52) \]

Clearly, the first and the third sums on the second line of eq. (S.51) are invariant under any \( SU(2N) \) rotation of the \( 2N \) real fields, but the second sum is more finicky — it is invariant only under rotations that preserve the \( E \) matrix, or rather the \( 2N \times 2N \) antisymmetric matrix

\[ \mathcal{E} = E_{2\times 2} \otimes 1_{N\times N} = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 \end{pmatrix} \]

Thus, due to the coupling of the \( N \) charged fields to the EM fields \( A^\mu \), the Lagrangian (16) does not have a global \( SO(2N) \) symmetry; instead, the symmetries are limited to the subgroup of \( SO(2N) \) that preserves the \( \mathcal{E} \) matrix.

That subgroup happens to be equivalent to \( U(N) \). This maybe hard to see in the language of \( 2N \) real fields, but it becomes obvious once we go back to complex fields that have definite electric charges (\( +q \) for all the \( \Phi^a \) and \( -q \) for all the \( \Phi_\alpha^* \)). In this basis, good symmetries should commute with the electric charge operators, so they should not mix the \( \Phi \) and \( \Phi^* \) fields with each other. On the other hand, any complex-linear mixing of the \( \Phi^a \) with each other (and of the \( \Phi_a^* \) with each other) is OK. As we saw in part (a), the symmetry group satisfying this condition — in addition to preserving the \( \sum_a |\Phi^a|^2 \) — is the unitary group \( U(N) \).
Problem 3(e):
Suppose an infinitesimal $SO(2N)$ transform (19) were a symmetry of the theory. Then the Noether current for this symmetry would be

$$J^\mu_C = \sum_a \left( \frac{\partial L}{\partial (\partial_\mu \Phi_a^*)} = D^\mu \Phi_a^* \right) \times \sum_b C^{ab} \Phi_b^* + \sum_a \left( \frac{\partial L}{\partial (\partial_\mu \Phi_a)} = D^\mu \Phi_a \right) \times \sum_b C^{*}_{ab} \Phi^b$$

$$= \sum_{a,b} \left( C^*_{ab} \Phi^b D^\mu \Phi^a + C^{ab} \Phi^*_b D^\mu \Phi^*_a \right)$$

$$= \frac{1}{2} \sum_{a,b} \left( C^*_{ab} I^{b,a} + C^{ab} I^{*}_{ba} \right)$$

$$= \frac{1}{2} \text{tr}(C^* I^\mu) + \frac{1}{2} \text{tr}(CI^{*\mu})$$  (S.54)

where $I^\mu_{ab}$ is the complex antisymmetric matrix of currents

$$I^{ab} = \Phi^b D^\mu \Phi^a - \Phi^a D^\mu \Phi^b = -I^\mu_{ab}$$  (S.55)

and $I^{*\mu}_{ab}$ is a similar antisymmetric matrix of the complex conjugate currents

$$I^{*\mu}_{ba} = \Phi^*_b D^\mu \Phi^*_a - \Phi^*_a D^\mu \Phi^*_b = -I^{\mu*}_{ab}.$$  (S.56)

These current are electrically charged — the $I^{\mu ab}$ have charge $+2q$ while the $I^{\mu*}_{ab}$ have charge $-2q$ — so let’s take their covariant divergences

$$D_\mu I^{ab} = \partial_\mu I^{ab} + 2iqA_\mu I^{ab} \quad \text{and} \quad D_\mu I^{*\mu}_{ab} = \partial_\mu I^{*\mu}_{ab} - 2iqA_\mu I^{\mu*}_{ab}.$$  (S.57)

Using the Leibniz rules for the $D_\mu$, we have

$$D_\mu I^{ab} = (D_\mu \Phi^a)(D^\mu \Phi^b) + \Phi^a(D_\mu D^\mu \Phi^b) - (a \leftrightarrow b)$$

$$= \Phi^a(D_\mu D^\mu \Phi^b) - \Phi^b(D_\mu D^\mu \Phi^a)$$

$$\langle \text{by equations of motion} \rangle$$

$$= \Phi^a \times \left( -m^2 - \frac{\lambda}{2} \sum |\Phi|^2 \right) \Phi^b - \Phi^b \times \left( -m^2 - \frac{\lambda}{2} \sum |\Phi|^2 \right) \Phi^a$$

$$= 0,$$  (S.58)

and likewise $D_\mu I^{\mu*}_{ab} = 0$. In light of eqs. (S.57) this means that the ordinary (non-covariant)
divergences of these currents do not vanish; instead

\[ \partial_\mu I^{\mu ab} = -2i q A_\mu I^{\mu ab}, \]
\[ \partial_\mu I^{\mu*}_{ab} = +2i q A_\mu I^{\mu*}_{ab}, \]

and the corresponding charges \( Q^{ab} = -Q^{ba} = \int d^3 x I^{0ab} \) and \( Q^*_{ab} = -Q^*_{ba} \) are NOT conserved.