Problem 1(b):
Let’s start with \([\hat{L}_i, \hat{x}_j]\). Since the second term in the angular momentum (5) commutes with the position operators, only the first term contributes to the commutator. Thus,

\[
[\hat{L}_i, \hat{x}_j] = [\hat{\mathbf{x}} \times \hat{\pi}, \hat{x}_j] = \epsilon_{ik\ell} [\hat{x}_k \hat{\pi}_\ell, \hat{x}_j] = -i\hbar \delta_{ij} + (\hat{\pi}_j, \hat{x}_j = 0) \hat{\pi}_\ell \tag{6.a}
\]

Next, consider the \([\hat{L}_i, \hat{\pi}_j]\) commutator. This time, the second term in the angular momentum (5) does not commute with the \(\hat{\pi}_j\); instead,

\[
[\hat{L}_i, \hat{\pi}_j] = \frac{-qM}{c} \frac{\hat{x}_i}{\hat{r}} = \frac{-qM}{c} \hat{x}_i \frac{\partial}{\partial \hat{x}_j} \frac{\hat{r}}{\hat{r}} = -i\hbar \epsilon_{ik\ell} \hat{x}_k = -i\hbar \epsilon_{ik\ell} \hat{x}_k \hat{x}_j + + i\hbar \epsilon_{ik\ell} \hat{x}_k \hat{x}_j. \tag{S.1}
\]

At the same time, the first term in the angular momentum (5) produces

\[
[\epsilon_{ik\ell} \hat{x}_k \hat{\pi}_\ell, \pi_j] = \epsilon_{ik\ell} [\hat{x}_k, \hat{\pi}_j] \hat{\pi}_\ell + \epsilon_{ik\ell} \hat{x}_k [\hat{\pi}_\ell, \hat{\pi}_j] = \epsilon_{ik\ell} (i\hbar \delta_{k j}) \hat{\pi}_\ell + \epsilon_{ik\ell} \hat{x}_k \times \frac{i\hbar qM}{c} \epsilon_{\ell j m} \frac{\hat{x}_m}{\hat{r}^3} \tag{S.2}
\]

Note that the second term on the bottom line here precisely cancels the RHS of eq. (S.1). Consequently, when we commute the net angular momentum (5) with the kinematic momentum \(\hat{\pi}_j\), we obtain

\[
[\hat{L}_i, \hat{\pi}_j] = (S.2) + (S.1) = i\hbar \epsilon_{ij\ell} \hat{\pi}_\ell + 0. \tag{6.b}
\]

Finally, let’s verify the third line of eq. (6). The first two lines of eq. (6) tell us that the \(\hat{L}_i\) act on the position and kinematic momentum operators as angular momenta. Consequently, they also act as angular momenta on any other vector operator \(\hat{\mathbf{V}}\) made from the \(\hat{x}\)’s and \(\hat{\pi}\)’s, thus

\[
[\hat{L}_i, \hat{\mathbf{V}}_j] = i\hbar \epsilon_{ijk} \hat{V}_k. \tag{S.3}
\]

The proof of this theorem is exactly similar to what you have done (or at least should have done) in the undergraduate school for the ordinary angular momentum \(\hat{\mathbf{J}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}} + \hat{\mathbf{S}}\), so I
would not repeat it here. All we need for now is the special case of this theorem for the \( \hat{L} \) vector itself — which is made from the positions and kinematic momenta operators according to eq. (5). Thus, for \( \hat{V} = \hat{L} \), eq. (S.3) becomes

\[
[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{K}_k. \tag{6.6c}
\]

Problem 1(c):
Commutation of the angular momentum (5) with the Hamiltonian (3) follows trivially from the commutation relations (6). In particularly, eq. (6.a) assures that the \( \hat{L}_i \) operators commute with the \( \hat{r}^2 = \hat{k}_k\hat{x}_k \) and hence with any function \( V(\hat{r}) \) of the radius operator \( \hat{r} \). Likewise, eq. (6.b) makes the angular momenta commute with the \( \vec{\pi}^2 = \hat{\pi}_j\hat{\pi}_j \). Consequently, a Hamiltonian of the form (3) commutes with the angular momenta (5).

Problem 1(d):
In the coordinate basis, \( \hat{\pi} = -i\hbar\vec{D} = -i\hbar\nabla - \frac{q}{c}\hat{A} \), hence

\[
\hat{L} = -i\hbar\vec{x} \times \nabla - \frac{q}{c}\vec{x} \times \vec{A} - \frac{qM}{c}\frac{\vec{x}}{r}. \tag{S.5}
\]

Let’s take a closer look at the second term here for the monopole’s vector potential (7):

\[
-\frac{q}{c}\vec{x} \times \vec{A} = -\frac{qM}{c}\frac{\pm 1 - \cos \theta}{r\sin \theta} (\vec{x} \times \vec{e}_\phi = -r\vec{e}_\theta) = \frac{qM}{c}\frac{\pm 1 - \cos \theta}{\sin \theta} \vec{e}_\theta. \tag{S.6}
\]

In particular, for the \( z \) component of the angular momentum, we have

\[
(\vec{e}_\theta)_z = -\sin \theta \tag{S.7}
\]

and therefore

\[
-\frac{q}{c}(\vec{x} \times \vec{A})_z = -\frac{qM}{c} \times (\pm 1 - \cos \theta) = +\frac{qM}{c} \times (\cos \theta - 1). \tag{S.8}
\]
At the same time, the third term on the RHS of eq. (S.5) has \( z \) component

\[
- \frac{qM}{c} \left( \frac{X}{r} \right)_z = - \frac{qM}{c} \times \cos \theta, \tag{S.9}
\]

while the \( z \) component of first term on the RHS of (S.5) is the usual

\[
- \imath \hbar (\mathbf{x} \times \nabla)_z = - \imath \hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)_{\text{Cartesian}} = - \imath \hbar \left( \frac{\partial}{\partial \phi} \right)_{\text{spherical}}. \tag{S.10}
\]

Finally, combining the three terms, we obtain

\[
\hat{L}_z = - \imath \hbar \frac{\partial}{\partial \phi} + \frac{qM}{c} \times (\cos \theta + 1) - \frac{qM}{c} \times \cos \theta = - \imath \hbar \frac{\partial}{\partial \phi} \mp \frac{qM}{c}. \tag{8}
\]

**Problem 1(e):**

Consider an eigenstate \( |\ell, m\rangle \) of the orbital angular momentum (5), that is, a simultaneous eigenstate of \( \hat{L}^2 \) and \( \hat{L}_z \) with eigenvalues \( \ell(\ell + 1)\hbar^2 \) and \( m\hbar \). In spherical coordinates, the \( \phi \) dependence of the eigenstate’s wavefunction \( \Psi_{\ell,m}(\theta, \phi) \) follows from the equation (8):

In the Northern hemisphere’s gauge,

\[
\Psi^N_{\ell,m}(\theta, \phi) = F_{\ell,m}(\theta) \times \exp \left( \imath \left( m + \frac{qM}{\hbar c} \right) \phi \right),
\]

In the Southern hemisphere’s gauge,

\[
\Psi^S_{\ell,m}(\theta, \phi) = F_{\ell,m}(\theta) \times \exp \left( \imath \left( m - \frac{qM}{\hbar c} \right) \phi \right).
\]

For convenience, let’s give the exponents here names

\[
k_{\pm} = m \pm \frac{qM}{\hbar c}. \tag{S.12}
\]

These exponents may be positive or negative, but for the orbital angular momentum, both \( k_{\pm} \) must be integers to assure single-valuedness of the wavefunctions (S.11). At the same time, the Dirac’s quantization condition for the electric and magnetic charges requires \( q \times M \) to be integer in units of \( \frac{1}{2} \hbar c \) and hence either integer or half integer in units of \( \hbar c \). Consequently,

\[
\text{for integer } \frac{qM}{\hbar c}, \quad m — \text{ and hence } \ell — \text{ must be integer,}
\]

\[
\text{but for half-integer } \frac{qM}{\hbar c}, \quad m — \text{ and hence } \ell — \text{ must be half-integer!}
\]

Now consider the \( \theta \) dependence of the spherical harmonics (S.11). For every complete \( \hat{L} \)-multiplet — i.e., for every fixed \( \ell \) — at least one of the harmonics should have a non-zero
value at the North pole $\theta = 0$. Also, in the Northern hemisphere’s gauge, that $\Psi^N_{\ell,m}$ should be non-singular at the pole. In terms of eqs. (S.11), this calls for

\[
\text{non-singular } \Psi^N_{\ell,m}(\theta, \phi) = F_{\ell,m}(\theta) \times \exp(ik_+\phi) \text{ at } \theta = 0 \text{ with } F(\theta = 0) \neq 0. \quad (S.14)
\]

Due to coordinate singularity at the North pole, this is possible only for the $k_+ = 0$ and therefore

\[
m = -\frac{qM}{\hbar c}.
\]

Likewise, for every $\ell$, at least one of the harmonics should have a non-zero value at the South pole $\theta = \pi$, hence

\[
\text{non-singular } \Psi^S_{\ell,m}(\theta, \phi) = F_{\ell,m}(\theta) \times \exp(ik_-\phi) \text{ at } \theta = \pi \text{ with } F(\theta = \pi) \neq 0, \quad (S.15)
\]

which calls for $k_- = 0$ and therefore

\[
m = +\frac{qM}{\hbar c}.
\]

Thus, any multiplet of $|\ell, m\rangle$ eigenstates of the angular momentum must contain states with $m = \pm qM/\hbar c$,

\[
\forall \ell \exists m = \pm \frac{qm}{\hbar c}. \quad (S.16)
\]

In the absence of a monopole, this means that every multiplet of the orbital angular momentum must contain an $|m = 0\rangle$ state — which is indeed true for every integer $\ell$. But in presence of a monopole, eq. (S.16) requires $\ell$ to be large enough, namely

\[
\ell \geq \frac{|qM|}{\hbar c}. \quad (S.17)
\]

Together with eq. (S.13), this means that the allowed values of $\ell$ run over

\[
\ell = \frac{|qM|}{\hbar c}, \frac{|qM|}{\hbar c} + 1, \frac{|qM|}{\hbar c} + 2, \ldots \quad (9)
\]
Problem 2(a):
In 3D notations, the Lagrangian (10) for the massive vector field is

\[ L = \frac{1}{2} (E^2 - B^2) + \frac{1}{2} m^2 (A_0^2 - A^2) - J_0 A_0 + J \cdot A \]

\[ = \frac{1}{2} (-\dot{A} - \nabla A_0)^2 + \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2} m^2 (A_0^2 - A^2) - J_0 A_0 + J \cdot A. \]  

(S.18)

Note that only the first term on the last line contains any time derivatives at all, and it does not contain the \( \dot{A}_0 \) but only the \( \dot{A} \). Consequently, \( \partial L/\partial \dot{A}_0 = 0 \) and the scalar potential \( A_0(x) \) does not have a canonical conjugate field. On the other hand, the vector potential \( A(x) \) does have a canonical conjugate, namely

\[ \frac{\delta L}{\delta \dot{A}(x)} = \left. \frac{\partial L}{\partial \dot{A}} \right|_{x} = -(-\dot{A}(x) - \nabla A_0(x)) = -E(x). \]  

(S.19)

Problem 2(b):
In terms of the Hamiltonian and Lagrangian densities, eq. (11) means

\[ \mathcal{H} = -\dot{\mathbf{A}} \cdot \mathbf{E} - L. \]  

\( 11' \)

Expressing all fields in terms \( A, E, \) and \( A_0 \), we have

\[ \dot{A} = -E - \nabla A_0, \]

\[ -\dot{A} \cdot E = E^2 + E \cdot \nabla A_0, \]  

(S.20)

and consequently,

\[ \mathcal{H} = \frac{1}{2} E^2 + E \cdot \nabla A_0 - \frac{1}{2} m^2 A_0^2 + A_0 J_0 + \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2} m^2 A^2 - A \cdot J. \]  

(S.21)

Taking the \( \int d^3x \) integral of this density and integrating by parts the \( E \cdot \nabla A_0 \) term, we arrive at the Hamiltonian (12). \( \textit{Q.E.D.} \)
Problem 2(c):
Evaluating the derivatives of $\mathcal{H}$ in eq. (13) gives us
\[
\frac{\delta H}{\delta A_0(x)} \equiv \frac{\partial \mathcal{H}}{\partial (A_0)} - \nabla_i \frac{\partial \mathcal{H}}{\partial (\nabla_i A_0)} = -m^2 A_0 + J_0 - \nabla_i E^i. \quad (S.22)
\]

If the scalar field $A_0$ had a canonical conjugate $\pi_0(x,t)$, its time derivative $\partial \pi_0/\partial t$ would be given by the right hand side of eq. (S.22). But the $A_0(x,t)$ does not have a canonical conjugate, so instead of a Hamilton equation of motion we have a time-independent constraint (13), namely
\[
m^2 A_0 = J_0 - \nabla \cdot E. \quad (S.23)
\]

In the massless EM case, a similar constraint gives rise to the Gauss Law $\nabla \cdot E = J_0$. But the massive vector field does not obey the Gauss Law; instead, eq. (S.23) gives us a formula for the scalar potential $A_0$ in terms of $\mathbf{E}$ and $J_0$.

However, Hamilton equations for the vector fields $\mathbf{A}$ and $\mathbf{E}$ are honest equations of motion. Specifically, evaluating the derivatives of $\mathcal{H}$ in the first eq. (14), we find
\[
\frac{\delta H}{\delta E^i(x)} \equiv \frac{\partial \mathcal{H}}{\partial (E^i)} - \nabla_j \frac{\partial \mathcal{H}}{\partial (\nabla_j E^i)} = E^i + \nabla_i A_0,
\]
which leads to Hamilton equation
\[
\frac{\partial}{\partial t} \mathbf{A}(x,t) = -\mathbf{E}(x,t) - \nabla A_0(x,t). \quad (S.24)
\]

Similarly, in the second eq. (14) we have
\[
\frac{\delta H}{\delta A^i(x)} \equiv \frac{\partial \mathcal{H}}{\partial (A^i)} - \nabla_j \frac{\partial \mathcal{H}}{\partial (\nabla_j A^i)} = m^2 A^i - J^i - \nabla_j (\epsilon^{jik}(\nabla \times A)^k)
\]
and hence Hamilton equation
\[
\frac{\partial}{\partial t} \mathbf{E}(x,t) = m^2 \mathbf{A} - \mathbf{J} + \nabla \times (\nabla \times \mathbf{A}). \quad (S.25)
\]
Problem 2(d):
In 3D notations, the Euler–Lagrange field equations (15) or $\partial_\mu F^{\mu\nu} + m^2 A^\nu = J^\nu$ become

\[
\nabla \cdot E + m^2 A_0 = J_0, \quad (S.26)
\]
\[
-\dot{E} + \nabla \times B + m^2 A = J, \quad (S.27)
\]

where

\[
E \overset{\text{def}}{=} -\dot{A} - \nabla A_0, \quad (S.28)
\]
\[
B \overset{\text{def}}{=} \nabla \times A. \quad (S.29)
\]

Clearly, eq. (S.26) is equivalent to eq. (S.23) while eq. (S.27) is equivalent to eq. (S.25) (provided $B$ is defined as in eq. (S.29)). Finally, eq. (S.28) is equivalent to eq. (S.24), although their origins differ: In the Lagrangian formalism, eq. (S.28) is the definition of the $E$ field in terms of $A_0$, $A$ and their derivatives, while in the Hamiltonian formalism, $E$ is an independent conjugate field and eq. (S.24) is the dynamical equation of motion for the $\dot{A}$.  

Problem 3:
Let start with the $[\hat{A}, \hat{H}]$ commutator. In light of eq. (18), we have

\[
[\hat{A}^i(x), \hat{H}] = \int d^3y \left[ \hat{A}^i(x), \left( \frac{1}{2} \dot{\hat{E}}^2 + \frac{1}{2m^2}(\dot{\hat{J}}_0 - \nabla \cdot \dot{\hat{E}})^2 + \frac{1}{2}(\nabla \times \dot{\hat{A}})^2 + \frac{1}{2}m^2 \dot{\hat{A}}^2 - \hat{J} \cdot \dot{\hat{A}} \right)(y) \right] \quad (S.30)
\]

where all operators are at the same time $t$ as the $\hat{A}^i(x,t)$. Since all the $\hat{A}^i(x)$ operators commute with each other at equal times, the last three terms in the Hamiltonian density do not contribute to the commutator (S.30). But for the first term we have

\[
[\hat{A}^i(x), \frac{1}{2} \dot{\hat{E}}^2(y)] = \frac{1}{2} \{ \dot{\hat{E}}^j(y), [\hat{A}^i(x), \dot{\hat{E}}^j(y)] \} = \frac{1}{2} \{ \dot{\hat{E}}^j(y), -i\delta^{ij}\delta^{(3)}(x-y) \} = -i\delta^{(3)}(x-y) \times \dot{\hat{E}}^i(y) \quad (S.31)
\]

while for the second term we have

\[
\left[ \hat{A}^i(x), \left( \hat{J}_0(y) - \nabla \cdot \dot{\hat{E}}(y) \right) \right] = 0 - \frac{\partial}{\partial y^i}[\hat{A}^i(x), \dot{\hat{E}}^j(y)] = +i\delta^{ij} \frac{\partial}{\partial y^j}\delta^{(3)}(x-y) \quad (S.32)
\]
and hence

\[
\left[ \hat{A}^i(x), \frac{1}{2m^2} \left( \hat{J}_0(y) - \nabla \hat{E}(y) \right)^2 \right] = \frac{1}{m^2} \left( \hat{J}_0(y) - \nabla \hat{E}(y) \right) \times + i \delta^{ij} \frac{\partial}{\partial y^j} \delta^{(3)}(x - y) = \hat{A}_0(y) \times i \frac{\partial}{\partial y^i} \delta^{(3)}(x - y)
\]

(S.33)

where the second equality follows from eq. (17). Plugging these all these commutators into eq. (S.30) and integrating over \( y \), we obtain

\[
\hat{A}^i(x) = \int d^3 y \left( -i \delta^{(3)}(x - y) \times \hat{E}^i(y) + \hat{A}_0(y) \times i \frac{\partial}{\partial y^i} \delta^{(3)}(x - y) + 0 + 0 + 0 \right)
\]

integrating by parts

\[
= \int d^3 y \left( -i \delta^{(3)}(x - y) \right) \times \left( \hat{E}^i(y) + \frac{\partial}{\partial y^i} \hat{A}_0(y) \right)
\]

\[
= -i \left( \hat{E}^i(x) + \frac{\partial}{\partial x^i} \hat{A}_0(x) \right).
\]

(S.34)

In other words, \( [\hat{A}(x), \hat{H}] = -i \hat{E}(x) - i \nabla \hat{A}_0(x) \) and consequently in the Heisenberg picture,

\[
\frac{\partial}{\partial t} \hat{A}(x, t) = -i [\hat{A}(x), \hat{H}] = -\hat{E}(x, t) - \nabla \hat{A}_0(x, t).
\]

(S.35)

Clearly, this Heisenberg equation is the quantum equivalent of the classical Hamilton equation (S.24).

Now consider the \( [\hat{E}, \hat{H}] \) commutator. Similarly to eq.(S.30), we have

\[
[\hat{E}^i(x, t), \hat{H}] = \int d^3 y \left[ \hat{E}^i(x, t), \left( \frac{1}{2} \hat{E}^2 + \frac{1}{2} m^2 \hat{A}_0^2 + \frac{1}{2} \hat{B}^2 + \frac{1}{2} m^2 \hat{A}^2 - \hat{J} \cdot \hat{A} \right)(y, t) \right]
\]

(S.36)

where \( m^2 \hat{A}_0 = \hat{J}_0 - \nabla \cdot \hat{E} \) according to eq. (17) and \( \hat{B} \overset{\text{def}}{=} \nabla \times \hat{A} \). At equal times the \( \hat{E}^i(x) \) operator commutes with all the \( \hat{E}^j(y) \) and hence with \( \hat{E}^2(y), \hat{A}_0(y), \) and \( \hat{A}_0^2(y) \); this eliminates the first two terms in the Hamiltonian density from the commutator (S.36). For
the remaining three terms we have

\[
\begin{align*}
\left[ \hat{E}^i(x), (-\hat{J} \cdot \hat{A})(y) \right] &= -\hat{J}^j(y) \times \left[ \hat{E}^i(x), \hat{A}^j(y) \right] \\
&= -\hat{J}^j(y) \times +i\delta^{ij}\delta^{(3)}(x - y) \\
&= -i\delta^{(3)}(x - y) \times \hat{J}^i(y), \\
\left[ \hat{E}^i(x), \frac{1}{2}m^2\hat{A}(y) \right] &= +im^2\delta^{(3)}(x - y) \times \hat{A}^i(y), \\
\left[ \hat{E}^i(x), \hat{B}^j(y) \right] &= e^{jkl} \frac{\partial}{\partial y^k} \left[ \hat{E}^i(x), \hat{A}^l(y) \right] \\
&= e^{jkl} \frac{\partial}{\partial y^k} \left( +i\delta^{il}\delta^{(3)}(x - y) \right) \\
&= +ie^{jki} \frac{\partial}{\partial y^k}\delta^{(3)}(x - y), \\
\left[ \hat{E}^i(x), \frac{1}{2}\hat{B}^2(y) \right] &= \hat{B}^i(y) \times +ie^{jki} \frac{\partial}{\partial y^k}\delta^{(3)}(x - y) \\
&= -ie^{jki} \frac{\partial}{\partial y^k}\hat{B}^i(y) \times \delta^{(3)}(x - y) + \text{a total derivative} \\
&= +i(\nabla \times \hat{B})^i(y) \times \delta^{(3)}(x - y) + \text{a total derivative}. \tag{S.37}
\end{align*}
\]

Thus

\[
\left[ \hat{E}(x), \hat{H}(y) \right] = i\delta^{(3)}(x - y) \times \left( \nabla \times \hat{B}(y) + m^2\hat{A}(y) - \hat{J}(y) \right) + \text{a total derivative}, \tag{S.38}
\]

hence

\[
\begin{align*}
\left[ \hat{E}(x), \hat{H} \right] &= \int d^3y \left( i\delta^{(3)}(x - y) \times \left( \nabla \times \hat{B}(y) + m^2\hat{A}(y) - \hat{J}(y) \right) + \text{a total derivative} \right) \\
&= i \left( \nabla \times \hat{B}(x) + m^2\hat{A}(x) - \hat{J}(x) \right), \tag{S.39}
\end{align*}
\]

and therefore in the Heisenberg picture

\[
\frac{\partial}{\partial t} \hat{E}(x, t) = -i \left[ \hat{E}(x, t), \hat{H} \right] = +\nabla \times \hat{B}(x) + m^2\hat{A}(x) - \hat{J}(x). \tag{S.40}
\]

Again, this Heisenberg equation is the quantum equivalent of the classical Hamilton equation (S.25).