Problem 1(a):
All the commutators in this question follow from the bosonic commutation relations (1) via the Leibniz rule:

\[ [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger] = [\hat{a}_\alpha^\dagger, \hat{a}_\gamma^\dagger] \hat{a}_\beta + \hat{a}_\alpha^\dagger \hat{a}_\gamma^\dagger = 0 + \delta_{\beta,\gamma} \hat{a}_\alpha^\dagger, \quad (S.1) \]

\[ [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\delta] = [\hat{a}_\alpha^\dagger, \hat{a}_\delta] \hat{a}_\beta + \hat{a}_\alpha^\dagger [\hat{a}_\beta, \hat{a}_\delta] = -\delta_{\alpha,\delta} \hat{a}_\beta + 0 = -\delta_{\alpha,\delta} \hat{a}_\beta, \quad (S.2) \]

\[ [\hat{a}_\alpha \hat{a}_\beta^\dagger, \hat{a}_\alpha^\dagger \hat{a}_\delta] = [\hat{a}_\alpha \hat{a}_\beta^\dagger, \hat{a}_\delta^\dagger] \hat{a}_\alpha + \hat{a}_\alpha \hat{a}_\beta^\dagger \hat{a}_\delta = \delta_{\beta,\gamma} \hat{a}_\alpha^\dagger \hat{a}_\delta - \delta_{\alpha,\delta} \hat{a}_\beta \hat{a}_\gamma^\dagger, \quad (S.3) \]

\[ [\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta] = [\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta] + \hat{a}_\alpha^\dagger [\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta] \hat{a}_\delta + \hat{a}_\alpha^\dagger \hat{a}_\mu^\dagger \hat{a}_\nu \hat{a}_\beta \hat{a}_\gamma \hat{a}_\delta + \hat{a}_\alpha^\dagger \hat{a}_\mu^\dagger \hat{a}_\nu \hat{a}_\beta \hat{a}_\gamma \hat{a}_\delta - \delta_{\nu\alpha} \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta \hat{a}_\mu^\dagger \hat{a}_\nu \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta. \quad (S.4) \]

Problem 1(b):
First, let’s prove by induction that for integer \( n \geq 0 \), \([\hat{a}, (\hat{a}^\dagger)^n] = n \times (\hat{a}^\dagger)^{n-1} \). The induction base is easy to check: For \( n = 0 \) we have \([\hat{a}, (\hat{a}^\dagger)^0] = [\hat{a}, 1] = 0 \times 0 \) whatever, while for \( n = 1 \) we have \([\hat{a}, (\hat{a}^\dagger)^1] = [a, \hat{a}^\dagger] = 1 \times (\hat{a}^\dagger)^0 \). Now suppose \([\hat{a}, (\hat{a}^\dagger)^n] = n(\hat{a}^\dagger)^{n-1} \) for some \( n \); then for \( n + 1 \) we have

\[ [\hat{a}, (\hat{a}^\dagger)^{n+1}] = [\hat{a}, (\hat{a}^\dagger)^n \times \hat{a}^\dagger] = [a, (\hat{a}^\dagger)^n] \times \hat{a}^\dagger + (\hat{a}^\dagger)^n \times [a, \hat{a}^\dagger] = n(\hat{a}^\dagger)^{n-1} \times \hat{a}^\dagger + (\hat{a}^\dagger)^n \times 1 = (n + 1) \times (\hat{a}^\dagger)^n. \quad (S.5) \]

Similarly, for any integer \( n \geq 0 \), \([\hat{a}^\dagger, (\hat{a}^\dagger)^n] = -n(\hat{a}^\dagger)^{n-1} \); again, the proof is by induction, which is so similar to the above that I don’t need to spell it out.

Next, consider an analytic function \( f \) of the creation operator. Analytic functions can be expanded into power series, \( f(x) = f_0 + f_1 x + f_2 x^2 + \cdots \); substituting \( x \mapsto \hat{a}^\dagger \) into such series for \( f \), we build the operator

\[ f(\hat{a}^\dagger) \overset{\text{def}}{=} \sum_{n=0}^{\infty} f_n \times (\hat{a}^\dagger)^n = f_0 + f_1 \times \hat{a}^\dagger + f_2 \times (\hat{a}^\dagger)^2 + \cdots. \quad (S.6) \]
Likewise, for \( f'(x) \overset{\text{def}}{=} \frac{df}{dx} = 0 + f_1 + 2f_2x + 3f_3x^2 + \cdots \) we have

\[
\begin{align*}
f'(\hat{a}^\dagger) &= \sum_{n=0}^{\infty} nf_n \times (\hat{a}^\dagger)^{n-1}. & (S.7)
\end{align*}
\]

Consequently,

\[
\begin{align*}
[\hat{a}, f(\hat{a}^\dagger)] &= \sum_{n=0}^{\infty} f_n \times [\hat{a}, (\hat{a}^\dagger)^n] = \sum_{n=0}^{\infty} f_n \times n \times (\hat{a}^\dagger)^{n-1} = f'(\hat{a}^\dagger). & (S.8)
\end{align*}
\]

Similarly, for an analytic function of the annihilation operator, \( f(\hat{a}) = f_0 + f_1 \times \hat{a} + f_2 \times (\hat{a})^2 + \cdots \), we have

\[
\begin{align*}
[\hat{a}^\dagger, f(\hat{a})] &= \sum_{n=0}^{\infty} f_n \times [\hat{a}^\dagger, (\hat{a})^n] = \sum_{n=0}^{\infty} f_n \times (-n) \times (\hat{a})^{n-1} = -f'(\hat{a}). & (S.9)
\end{align*}
\]

\textbf{Q.E.D.}

\textbf{Problem 1(c):}

In light of part (b), \( [\hat{a}, \exp(c\hat{a}^\dagger)] = \exp'(c\hat{a}^\dagger) = c \exp(c\hat{a}^\dagger) \) and \( [\hat{a}^\dagger, \exp(c\hat{a})] = -\exp'(c\hat{a}) = -c \exp(c\hat{a}) \). Consequently,

\[
\begin{align*}
e^{-c\hat{a}} \hat{a}^\dagger e^{-c\hat{a}} &= \left( \hat{a}^\dagger e^{-c\hat{a}} - [\hat{a}^\dagger, e^{-c\hat{a}}] \right) e^{-c\hat{a}} = \left( \hat{a}^\dagger e^{-c\hat{a}} - (-c)e^{-c\hat{a}} \right) e^{-c\hat{a}} = \hat{a}^\dagger + c. & (S.10)
\end{align*}
\]

and likewise

\[
\begin{align*}
e^{-c\hat{a}} \hat{a}^\dagger e^{-c\hat{a}^\dagger} &= \left( \hat{a} e^{-c\hat{a}^\dagger} - [\hat{a}, e^{-c\hat{a}^\dagger}] \right) e^{-c\hat{a}^\dagger} = \left( \hat{a} e^{-c\hat{a}^\dagger} - (+c)e^{-c\hat{a}^\dagger} \right) e^{-c\hat{a}^\dagger} = \hat{a} - c. & (S.11)
\end{align*}
\]

Now, for any two operators \( \hat{X} \) and \( \hat{Y} \),

\[
\begin{align*}
\left( e^{\hat{X}} \hat{Y} e^{-\hat{X}} \right)^n &= e^{\hat{X}} \hat{Y} e^{-\hat{X}} \times e^{\hat{X}} \hat{Y} e^{-\hat{X}} \times \cdots \times e^{\hat{X}} \hat{Y} e^{-\hat{X}} = e^{\hat{X}} \hat{Y} \times \hat{Y} \times \cdots \hat{Y} e^{-\hat{X}} = e^{\hat{X}} \hat{Y}^n e^{-\hat{X}}. & (S.12)
\end{align*}
\]
Consequently, for any analytic function \( f(\hat{Y}) = f_0 + f_1 \hat{Y} + f_2 \hat{Y} + \cdots \),

\[
f\left( e^{\hat{X} \hat{Y} e^{-\hat{X}}} \right) = \sum_n f_n \left( e^{\hat{X} \hat{Y} e^{-\hat{X}}} \right)^n = \sum_n f_n \times e^{\hat{X} \hat{Y} e^{-\hat{X}}} = e^{\hat{X} \left( \sum_n f_n \hat{Y}^n \right) e^{-\hat{X}}} = e^{\hat{X} f(\hat{Y}) e^{-\hat{X}}}. \tag{S.13}
\]

In particular, for \( \hat{X} = c \hat{a} \) and \( \hat{Y} = \hat{a}^\dagger \),

\[
e^{c \hat{a}} f(\hat{a}^\dagger) e^{-c \hat{a}} = f\left( e^{c \hat{a} \hat{a}^\dagger e^{-c \hat{a}}} \right) = f(\hat{a}^\dagger + c), \tag{S.14}
\]

and likewise, for \( \hat{X} = c \hat{a}^\dagger \) and \( \hat{Y} = \hat{a} \),

\[
e^{c \hat{a}^\dagger} f(\hat{a}) e^{-c \hat{a}^\dagger} = f\left( e^{c \hat{a}^\dagger \hat{a} e^{-c \hat{a}^\dagger}} \right) = f(\hat{a} - c). \tag{S.15}
\]

\[Q.E.D.\]

Problem 1(d):

Since all the creation operators commute with each other, we may decompose any analytic function of multiple creation operators into a power series with respect to any particular \( \hat{a}_\alpha^\dagger \) as

\[
f(\text{multiple } \hat{a}^\dagger) = \sum_n F_n(\text{other } \hat{a}_\beta^\dagger) \times (\hat{a}_\alpha^\dagger)^n \tag{S.16}
\]

where \( F_n \) are some analytic functions of the *other* creation operators \( \hat{a}_\beta^\dagger \neq \hat{a}_\alpha^\dagger \). The same \( F_n \) appear in the partial derivative of \( f(\hat{a}^\dagger) \) with respect to the \( \hat{a}_\alpha^\dagger \),

\[
\frac{\partial f(\text{multiple } \hat{a}^\dagger)}{\partial \hat{a}_\alpha^\dagger} = \sum_n n \times F_n(\text{other } \hat{a}_\beta^\dagger) \times (\hat{a}_\alpha^\dagger)^{n-1}. \tag{S.17}
\]

Note that the creation operators \( \hat{a}_\beta^\dagger \) with \( \beta \neq \alpha \) commute with the \( \hat{a}_\alpha \) annihilation operator,
hence any function of such $\hat{a}^\dagger_{\beta \neq \alpha}$ also commutes with the $\hat{a}_\alpha$,

$$\left[ \hat{a}_\alpha, F_n(\text{other } \hat{a}_\beta^\dagger) \right] = 0,$$  \hspace{1cm} \text{(S.18)}

therefore

$$\left[ \hat{a}_\alpha, f(\text{multiple } \hat{a}^\dagger) \right] = \sum_n F_n(\text{other } \hat{a}_\beta^\dagger) \times \left[ \hat{a}_\alpha, (\hat{a}_\alpha^\dagger)^n \right]$$

$$= \sum_n F_n(\text{other } \hat{a}_\beta^\dagger) \times n(\hat{a}_\alpha^\dagger)^{n-1} = \frac{\partial f(\text{multiple } \hat{a}^\dagger)}{\partial \hat{a}_\alpha}. \hspace{1cm} \text{(S.19)}$$

This proves the first equation (4).

Similarly, any analytic function of multiple annihilation operators $\hat{a}_\beta$ — which also commute with each other — may be decomposed into a power series in any particular $\hat{a}_\alpha$ as

$$f(\text{multiple } \hat{a}) = \sum_n F_n(\text{other } \hat{a}_\beta) \times (\hat{a}_\alpha)^n$$  \hspace{1cm} \text{(S.20)}

where the $F_n$ are analytic functions of the remaining annihilation operators $\hat{a}_{\beta \neq \alpha}$ but not of the $\hat{a}_\alpha$ itself. Consequently, as operators all the $F_n(\text{other } \hat{a}_\beta)$ commute with the $\hat{a}_\alpha^\dagger$ and hence

$$\left[ \hat{a}_\alpha^\dagger, f(\text{multiple } \hat{a}) \right] = \sum_n F_n(\text{other } \hat{a}_\beta) \times \left[ \hat{a}_\alpha^\dagger, (\hat{a}_\alpha)^n \right]$$

$$= \sum_n F_n(\text{other } \hat{a}_\beta) \times -n(\hat{a}_\alpha)^{n-1} = -\frac{\partial f(\text{multiple } \hat{a})}{\partial \hat{a}_\alpha}. \hspace{1cm} \text{(S.21)}$$

This proves the second equation (4).

Now let’s proceed similarly to part (c). Applying the first two eqs. (4) to $f(\text{multiple } x) = \exp \left( \sum_\beta c_\beta x_\beta \right)$, we have

$$\left[ \hat{a}_\alpha, \exp \left( \sum_\beta c_\beta \hat{a}_\beta^\dagger \right) \right] = +\frac{\partial}{\partial \hat{a}_\alpha} \exp \left( \sum_\beta c_\beta \hat{a}_\beta^\dagger \right) = +c_\alpha \times \exp \left( \sum_\beta c_\beta \hat{a}_\beta^\dagger \right),$$

$$\left[ \hat{a}_\alpha^\dagger, \exp \left( \sum_\beta c_\beta \hat{a}_\beta \right) \right] = -\frac{\partial}{\partial \hat{a}_\alpha} \exp \left( \sum_\beta c_\beta \hat{a}_\beta \right) = -c_\alpha \times \exp \left( \sum_\beta c_\beta \hat{a}_\beta \right), \hspace{1cm} \text{(S.22)}$$
and consequently
\[
\exp\left(\sum c^\beta \hat{a}^\beta\right) \times \hat{a}_\alpha \times \exp\left(-\sum c^\beta \hat{a}^\beta\right) = \hat{a}_\alpha + c_\alpha,
\]
\[
\exp\left(\sum c^\beta \hat{a}^\dagger_\beta\right) \times \hat{a}_\alpha \times \exp\left(-\sum c^\beta \hat{a}^\dagger_\beta\right) = \hat{a}_\alpha - c_\alpha.
\]  
(S.23)

Finally, applying eq. (S.13) to these formulae, we obtain the last two eqs. (4) for any analytic function \( f \).  
Q.E.D.

Problem 2(a):

Classically, for each scalar field \( \Phi_a(x, t) \) there is a canonically conjugate field
\[
\Pi_a(x, t) = \frac{\delta L}{\delta \Phi_a(x)} \bigg|_{t} = \dot{\Phi}_a(x, t).
\]  
(S.24)

Consequently, the classical Hamiltonian density is
\[
\mathcal{H} = \sum_a \Pi_a \dot{\Phi}_a - L = \frac{1}{2} \sum_a \Pi_a^2 + \frac{1}{2} \sum_a (\nabla \Phi_a)^2 + \frac{m^2}{2} \sum_a \Phi_a^2 + \frac{\lambda}{24} \left(\sum_a \dot{\Phi}_a^2\right)^2
\]  
while the Poisson brackets involve \( \sum_a \) as well as \( \int d^3x \):
\[
[[A, B]] = \int d^3x \sum_a \left(\frac{\delta A}{\delta \Phi_a(x)} \frac{\delta B}{\delta \Pi_a(x)} - \frac{\delta A}{\delta \Pi_a(x)} \frac{\delta B}{\delta \Phi_a(x)}\right).
\]  
(S.26)

In particular,
\[
[[\Phi_a(x), \Phi_b(y)]] = 0, \quad [[\Pi_a(x), \Pi_b(y)]] = 0, \quad [[\Phi_a(x), \Pi_b(y)]] = \delta_{ab}\delta^{(3)}(x - y).
\]  
(S.27)

Consequently, in the quantum theory the corresponding quantum fields \( \hat{\Phi}_a(x, t) \) and \( \hat{\Pi}_a(x, t) \) obey similar equal-time commutation relations:
\[
[\hat{\Phi}_a(x, t), \hat{\Phi}_b(y, \text{same } t)] = 0,
\]
\[
[\hat{\Pi}_a(x, t), \hat{\Pi}_b(y, \text{same } t)] = 0,
\]
\[
[\hat{\Phi}_a(x, t), \hat{\Pi}_b(y, \text{same } t)] = i\delta_{ab}\delta^{(3)}(x - y).
\]  
(S.28)

And the Hamiltonian operator of the quantum theory follows from the classical Hamilto-
\[ \hat{H} = \int d^3x \hat{\mathcal{H}}(x, t) \quad \text{where} \]
\[ \hat{\mathcal{H}}(x, t) = \frac{1}{2} \sum_a \hat{\Pi}_a^2(x, t) + \frac{1}{2} \sum_a (\nabla \hat{\Phi}_a(x, t))^2 + \frac{m^2}{2} \sum_a \hat{\Phi}_a^2(x, t) + \frac{\lambda}{24} \left( \sum_a \hat{\Phi}_a^2(x, t) \right)^2. \]

(S.29)

Problem 2(b):
Applying the Leibniz rule to the equal-time commutators (S.28), we have
\[
\left[ \hat{\Phi}_a(y, t) \hat{\Pi}_b(y, t), \hat{\Phi}_c(x, t) \right] = \hat{\Phi}_a(y, t) \left[ \hat{\Pi}_b(y, t), \hat{\Phi}_c(x, t) \right] + \left[ \hat{\Phi}_a(y, t), \hat{\Phi}_c(x, t) \right] \hat{\Pi}_b(y, t)
\]
\[= \hat{\Phi}_a(y, t) \times (-i) \delta_{bc} \delta^{(3)}(y - x) + 0 \times \hat{\Pi}_b(y, t)\]
\[= -i \delta_{bc} \hat{\Phi}_a(y) \times \delta^{(3)}(y - x) \quad \text{(S.30)}\]

and likewise
\[
\left[ \hat{\Phi}_b(y, t) \hat{\Pi}_a(y, t), \hat{\Phi}_c(x, t) \right] = -i \delta_{ac} \hat{\Phi}_b(y, t) \times \delta^{(3)}(y - x). \quad \text{(S.31)}
\]

Hence, for the net charge operator \( \hat{Q}_{ab} \) as in eq. (4.4),
\[
\left[ \hat{Q}_{ab}(t), \hat{\Phi}_c(x, t) \right] = \int d^3y \left[ \hat{\Phi}_a(y, t) \hat{\Pi}_b(y, t) - \hat{\Phi}_b(y, t) \hat{\Pi}_a(y, t), \hat{\Phi}_c(x, t) \right]
\]
\[= \int d^3y \left( -i \delta_{bc} \hat{\Phi}_a(y, t) + i \delta_{ac} \hat{\Phi}_b(y, t) \right) \times \delta^{(3)}(y - x) \quad \text{(S.32)}
\]
\[= -i \delta_{bc} \hat{\Phi}_a(x, t) + i \delta_{ac} \hat{\Phi}_b(x, t). \]

Similarly,
\[
\left[ \hat{\Phi}_a(y, t) \hat{\Pi}_b(y, t), \hat{\Pi}_c(x, t) \right] = \hat{\Phi}_a(y, t) \left[ \hat{\Pi}_b(y, t), \hat{\Pi}_c(x, t) \right] + \left[ \hat{\Phi}_a(y, t), \hat{\Pi}_c(x, t) \right] \hat{\Pi}_b(y, t)
\]
\[= \hat{\Phi}_a(y, t) \times 0 + i \delta_{ac} \delta^{(3)}(y - x) \times \hat{\Pi}_b(y, t)\]
\[= +i \delta_{ac} \hat{\Pi}_b(y, t) \times \delta^{(3)}(y - x) \quad \text{(S.33)}
\]
and likewise
\[
\left[ \hat{\Phi}_b(y, t) \hat{\Pi}_a(y, t), \hat{\Pi}_c(x, t) \right] = +i \delta_{bc} \hat{\Pi}_a(y, t) \times \delta^{(3)}(y - x), \quad \text{(S.34)}
\]
hence
\[
\left[ \hat{Q}_{ab}(t), \hat{\Pi}_c(x) \right] = \int d^3y \left[ \hat{\Phi}_a(y, t)\hat{\Pi}_b(y, t) - \hat{\Phi}_b(y, t)\hat{\Pi}_a(y, t), \hat{\Pi}_c(x, t) \right] \\
= \int d^3y \left( +i\delta_{ac}\hat{\Pi}_b(y, t) - i\delta_{bc}\hat{\Pi}_a(y, t) \right) \times \delta^{(3)}(y - x) \\
= -i\delta_{bc}\hat{\Pi}_a(x, t) + \delta_{ac}\hat{\Pi}_b(x, t).
\] (S.35)

\[Q.E.D.\]

**Problem 2(c):**

The Hamiltonian operator (S.29) is $SO(N)$ invariant — in fact, each of the 4 terms comprising the Hamiltonian density $\hat{H}(x, t)$ is separately $SO(N)$ invariant — and that makes them commute with all the $\hat{Q}_{ab}$ charges. Indeed, suppose some $N$ operators $\hat{V}_c$ — which could be $\hat{\Phi}_c(x)$, or $\hat{\Pi}_c(x)$, or whatever — satisfy commutation relations similar to eqs. (4.5), namely

\[
\left[ \hat{Q}_{ab}(t), \hat{V}_c(same \ t) \right] = -i\delta_{bc}\hat{V}_a(t) + i\delta_{ac}\hat{V}_b(t),
\] (S.36)

then the $\sum_c \hat{V}_c^2$ operator commutes with all the charges $\hat{Q}_{ab}$ (at equal times). Here is the proof:

\[
\left[ \hat{Q}_{ab}, \sum_c \hat{V}_c^2 \right] = \sum_c \left[ \hat{Q}_{ab}, \hat{V}_c^2 \right] = \sum_c \left\{ \hat{V}_c, \left[ \hat{Q}_{ab}, \hat{V}_c \right] \right\} \\
= \sum_c \left\{ \hat{V}_c, \left( -i\delta_{bc}\hat{V}_a + i\delta_{ac}\hat{V}_b \right) \right\} \\
= -i \left\{ \hat{V}_b, \hat{V}_a \right\} + i \left\{ \hat{V}_a, \hat{V}_b \right\} \\
= 0.
\] (S.37)

In particular, letting $\hat{V}_c = \hat{\Pi}_c(x)$, or $\hat{V}_c = \hat{\Phi}_c(x)$, or $\hat{V}_c = \nabla\hat{\Phi}_c(x)$ — which also satisfy

\[
\left[ \hat{Q}_{ab}(t), \nabla\hat{\Phi}_c(x, t) \right] = \nabla \left[ \hat{Q}_{ab}(t), \hat{\Phi}_c(x, t) \right] = -i\delta_{bc} \nabla\hat{\Phi}_a(x, t) + i\delta_{ac} \nabla\hat{\Phi}_b(x, t)
\] (S.38)

— we immediately obtain

\[
\left[ \hat{Q}_{ab}(t), \sum_c \hat{\Pi}_c^2(x, t) \right] = 0, \quad \left[ \hat{Q}_{ab}(t), \sum_c \nabla\hat{\Phi}_c^2(x, t) \right] = 0, \quad \left[ \hat{Q}_{ab}(t), \sum_c \hat{\Phi}_c^2(x, t) \right] = 0.
\] (S.39)
hence also

$$\left[ \hat{Q}_{ab}(t), \left( \sum_c \hat{\Phi}_c^2(x, t) \right)^2 \right] = 0,$$

(S.40)

and therefore $$\left[ \hat{Q}_{ab}(t), \hat{H} \right] = 0.$$  \textit{Q.E.D.}

Problem 2(d):
The commutations relations (10) between the charges follow from expanding $$\hat{Q}_{cd}$$ into quantum fields according to eq. (8) and then using the commutators (9) of those fields with the $$\hat{Q}_{ab}$$ charge:

$$\left[ \hat{Q}_{ab}, \hat{Q}_{cd} \right] = \left[ \hat{Q}_{ab}, \int d^3x \left( \hat{\Phi}_c(x) \tilde{\Pi}_d(x) - \hat{\Phi}_d(x) \tilde{\Pi}_c(x) \right) \right]$$

$$= \int d^3x \left[ \hat{Q}_{ab}, \left( \hat{\Phi}_c(x) \tilde{\Pi}_d(x) - \hat{\Phi}_d(x) \tilde{\Pi}_c(x) \right) \right]$$

$$= \int d^3x \left( \hat{\Phi}_c(x) \left[ \hat{Q}_{ab}, \tilde{\Pi}_d(x) \right] + \left[ \hat{Q}_{ab}, \hat{\Phi}_c(x) \right] \tilde{\Pi}_d(x) \right.$$

$$\left. - \hat{\Phi}_d(x) \left[ \hat{Q}_{ab}, \tilde{\Pi}_c(x) \right] - \left[ \hat{Q}_{ab}, \hat{\Phi}_d(x) \right] \tilde{\Pi}_c(x) \right)$$

$$= \int d^3x \left( \hat{\Phi}_c \left( -i \delta_{bd} \tilde{\Pi}_a + i \delta_{ad} \tilde{\Pi}_b \right) + \left( -i \delta_{bc} \hat{\Phi}_a + i \delta_{ac} \hat{\Phi}_b \right) \tilde{\Pi}_d \right.$$

$$\left. - \hat{\Phi}_d \left( -i \delta_{bc} \tilde{\Pi}_a + i \delta_{ac} \tilde{\Pi}_b \right) - \left( -i \delta_{bd} \hat{\Phi}_a + i \delta_{ad} \hat{\Phi}_b \right) \tilde{\Pi}_c \right) \right) \@x$$

$$= -i \delta_{bd} \times \int d^3x \left( \hat{\Phi}_c \tilde{\Pi}_a - \hat{\Phi}_a \tilde{\Pi}_c \right) \@x + i \delta_{ad} \times \int d^3x \left( \hat{\Phi}_c \tilde{\Pi}_b - \hat{\Phi}_b \tilde{\Pi}_c \right) \@x$$

$$+ i \delta_{bc} \times \int d^3x \left( \hat{\Phi}_d \tilde{\Pi}_a - \hat{\Phi}_a \tilde{\Pi}_d \right) \@x - i \delta_{ac} \times \int d^3x \left( \hat{\Phi}_d \tilde{\Pi}_b - \hat{\Phi}_b \tilde{\Pi}_d \right) \@x$$

$$= -i \delta_{bd} \times \hat{Q}_{ca} + i \delta_{ad} \times \hat{Q}_{cb} + i \delta_{bc} \times \hat{Q}_{da} - i \delta_{ac} \times \hat{Q}_{db}$$

$$= -i \delta_{bc} \times \hat{Q}_{ad} + i \delta_{ac} \times \hat{Q}_{bd} + i \delta_{bd} \times \hat{Q}_{ac} - i \delta_{ad} \times \hat{Q}_{bc}.$$  \textit{Q.E.D.}

(S.41)

Note: since the charges are time independent, the fields in the above formulae may be evaluated at any time $$t$$, as long as it’s the same time for all the operators.
Problem 2(e):
In class, we have expanded a single free scalar fields \( \Phi(x) \) and its canonical conjugate \( \Pi(x) \) into creation and annihilation operators \( \hat{a}_p \) and \( \hat{a}^\dagger_p \). In the present \( N \)-field case, we may proceed exactly like in class, except that the creation and annihilation operators are labeled by the species index \( a = 1, \ldots, N \) in addition to the momentum mode \( p \). Thus,

\[
\hat{\Phi}_{a,p} = \sqrt{\frac{1}{2E_p}} (\hat{a}_{a,p} + \hat{a}^\dagger_{a,-p}), \quad \hat{\Pi}_{a,p} = \sqrt{\frac{E_p}{2}} (-i\hat{a}_{a,p} + i\hat{a}^\dagger_{a,-p}) \tag{S.42}
\]

— cf. eq. (14) of my notes — hence Fourier-transforming back to the coordinate space we get

\[
\hat{\Phi}_a(x) = \sum_p L^{-3/2} e^{ipx} \hat{\Phi}_{a,p} = L^{-3/2} \sum_p \sqrt{\frac{1}{2E_p}} e^{ipx} (\hat{a}_{a,p} + \hat{a}^\dagger_{a,-p}),
\]

\[
\hat{\Pi}_a(x) = \sum_p L^{-3/2} e^{ipx} \hat{\Pi}_{a,p} = L^{-3/2} \sum_p \sqrt{E_p} e^{ipx} (-i\hat{a}_{a,p} + i\hat{a}^\dagger_{a,-p}) \tag{S.43}
\]

\[
= \sum_k L^{-3/2} e^{-ikx} \hat{\Pi}_{a,k}^\dagger = L^{-3/2} \sum_p \sqrt{\frac{E_k}{2}} e^{-ikx} (i\hat{a}^\dagger_{a,k} - i\hat{a}_{a,-k}).
\]

Given this expansion of the quantum fields, we may expand integrals of fields bilinears into sums of \( \hat{a}\hat{a}, \hat{a}\hat{a}^\dagger, \hat{a}^\dagger\hat{a}, \) and \( \hat{a}^\dagger\hat{a}^\dagger \) operators. In particular,

\[
\int d^3 x \hat{\Phi}_a(x) \hat{\Pi}_b(x) = \int d^3 x L^{-3} \sum_{p,k} \sqrt{\frac{E_k}{4E_p}} \times e^{+ipx-ikx} \times (\hat{a}_{a,p} + \hat{a}^\dagger_{a,-p})(i\hat{a}^\dagger_{b,k} - i\hat{a}_{b,-k}) \times
\]

\[
\times \left( L^{-3} \int d^3 x e^{+ipx-ikx} = \delta_{p,k} \right) \tag{S.44}
\]

\[
= \sum_p \sqrt{\frac{1}{4}} (\hat{a}_{a,p} + \hat{a}^\dagger_{a,-p})(i\hat{a}^\dagger_{b,p} - i\hat{a}_{b,-p})
\]

\[
= \frac{i}{2} \sum_p \hat{a}_{a,p} \hat{a}^\dagger_{b,p} + \frac{i}{2} \sum_p \hat{a}^\dagger_{a,-p} \hat{a}^\dagger_{b,p} - \frac{i}{2} \sum_p \hat{a}_{a,p} \hat{a}_{b,-p} - \frac{i}{2} \sum_p \hat{a}^\dagger_{a,-p} \hat{a}_{b,-p}.
\]
Likewise,

\[
\int d^3 \mathbf{x} \hat{\Phi}_b(\mathbf{x}) \hat{\Pi}_a(\mathbf{x}) = \\
= \frac{i}{2} \sum_p \hat{a}_{b,p} \hat{a}^\dagger_{a,p} + \frac{i}{2} \sum_p \hat{\bar{a}}_{b,-p} \hat{a}^\dagger_{a,-p} - \frac{i}{2} \sum_p \hat{\bar{a}}_{b,p} \hat{a}_{a,-p} - \frac{i}{2} \sum_p \hat{a}^\dagger_{b,-p} \hat{\bar{a}}_{a,-p} \quad (S.45)
\]

where in the two middle sums on the last line I have changed \( p \to -p \). Combining eqs. (S.44) and (S.45) we immediately obtain the expansion of the charge operators (8):

\[
\hat{Q}_{ab} = \int d^3 \mathbf{x} \hat{\Phi}_a(\mathbf{x}) \hat{\Pi}_b(\mathbf{x}) - \int d^3 \mathbf{x} \hat{\Phi}_b(\mathbf{x}) \hat{\Pi}_a(\mathbf{x}) \\
= \frac{i}{2} \sum_p \hat{a}_{a,p} \hat{a}^\dagger_{b,p} + \frac{i}{2} \sum_p \hat{a}^\dagger_{a,-p} \hat{a}^\dagger_{b,-p} - \frac{i}{2} \sum_p \hat{\bar{a}}_{a,p} \hat{\bar{a}}_{b,p} - \frac{i}{2} \sum_p \hat{\bar{a}}_{a,-p} \hat{\bar{a}}_{b,-p} \\
- \frac{i}{2} \sum_p \hat{\bar{a}}_{b,p} \hat{\bar{a}}_{a,p} - \frac{i}{2} \sum_p \hat{a}_{b,-p} \hat{\bar{a}}_{a,-p} + \frac{i}{2} \sum_p \hat{a}_{b,-p} \hat{\bar{a}}_{a,-p} + \frac{i}{2} \sum_p \hat{\bar{a}}_{b,-p} \hat{a}_{a,-p} \quad (S.46)
\]

where the commutators on the last line vanish. Hence

\[
\hat{Q}_{ab} = \frac{i}{2} \sum_p (\hat{a}_{a,p} \hat{a}^\dagger_{b,p} - \hat{\bar{a}}_{b,p} \hat{\bar{a}}^\dagger_{a,p}) - \frac{i}{2} \sum_p (\hat{\bar{a}}_{a,-p} \hat{\bar{a}}_{b,-p} - \hat{a}^\dagger_{b,-p} \hat{a}_{a,-p}) \\
\langle \text{changing } p \to -p \text{ in the second sum} \rangle \\
= \frac{i}{2} \sum_p (\hat{a}_{a,p} \hat{a}^\dagger_{b,p} - \hat{\bar{a}}_{b,p} \hat{\bar{a}}^\dagger_{a,p}) - \frac{i}{2} \sum_p (\hat{\bar{a}}_{a,p} \hat{\bar{a}}_{b,p} - \hat{a}^\dagger_{b,p} \hat{a}_{a,p}) \quad (S.47)
\]

\[
= \frac{i}{2} \sum_p (\{\hat{a}_{a,p} \hat{\bar{a}}^\dagger_{b,p}\} - \{\hat{\bar{a}}_{a,p} \hat{a}^\dagger_{b,p}\}).
\]

In each term in the last sum hear,

\[
\{\hat{a}_{a,p} \hat{\bar{a}}^\dagger_{b,p}\} = 2\hat{\bar{a}}^\dagger_{b,p} \hat{a}_{a,p} + [\hat{a}_{a,p} \hat{\bar{a}}^\dagger_{b,p}] = 2\hat{\bar{a}}^\dagger_{b,p} \hat{a}_{a,p} + \delta_{ab} \quad (S.48)
\]

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and likewise
\[
\{\hat{a}_{a,p}\hat{a}^\dagger_{b,p}\} = 2\hat{a}^\dagger_{b,p}\hat{a}_{a,p} + \delta_{ab},
\] (S.49)

hence
\[
\{\hat{a}_{a,p}\hat{a}^\dagger_{b,p}\} - \{\hat{a}_{a,p}\hat{a}^\dagger_{b,p}\} = 2\hat{a}^\dagger_{b,p}\hat{a}_{a,p} - 2\hat{a}^\dagger_{a,p}\hat{a}_{b,p}
\] (S.50)

and therefore
\[
\hat{Q}_{ab} = \sum_p (-i\hat{a}^\dagger_{a,p}\hat{a}_{b,p} + i\hat{a}^\dagger_{b,p}\hat{a}_{a,p}).
\] (11)

Q.E.D.

Problem 2(f):

In light of eq. (S.3),
\[
[\hat{a}^\dagger_{a,p}\hat{a}_{b,p}, \hat{a}^\dagger_{c,p}\hat{a}_{d,p}] = \delta_{p,p'}(\delta_{bc}\hat{a}^\dagger_{a,p}\hat{a}_{d,p} - \delta_{ad}\hat{a}^\dagger_{c,p}\hat{a}_{b,p}).
\] (S.51)

Consequently, expanding the charges $\hat{Q}_{ab}$ and $\hat{Q}_{cd}$ according to eq. (11),
\[
[\hat{Q}_{ab}, \hat{Q}_{cd}] = \sum_{p,p'} \left( -[\hat{a}^\dagger_{a,p}\hat{a}_{b,p}, \hat{a}^\dagger_{c,p}\hat{a}_{d,p}] + [\hat{a}^\dagger_{b,p}\hat{a}_{a,p}, \hat{a}^\dagger_{c,p}\hat{a}_{d,p}] \\
+ [\hat{a}^\dagger_{a,p}\hat{a}_{b,p}, \hat{a}^\dagger_{d,p}\hat{a}_{c,p}] - [\hat{a}^\dagger_{b,p}\hat{a}_{a,p}, \hat{a}^\dagger_{d,p}\hat{a}_{c,p}] \right) \\
\left\{ \text{using eq. (S.51)} \right\}
\]
\[
= \sum_{p,p'} \delta_{p,p'} \left( -\left(\delta_{bc}\hat{a}^\dagger_{a,p}\hat{a}_{d,p} - \delta_{ad}\hat{a}^\dagger_{c,p}\hat{a}_{b,p}\right) + \left(\delta_{ac}\hat{a}^\dagger_{b,p}\hat{a}_{d,p} - \delta_{bd}\hat{a}^\dagger_{c,p}\hat{a}_{a,p}\right) \\
+ \left(\delta_{bd}\hat{a}^\dagger_{a,p}\hat{a}_{c,p} - \delta_{ac}\hat{a}^\dagger_{d,p}\hat{a}_{b,p}\right) - \left(\delta_{ad}\hat{a}^\dagger_{b,p}\hat{a}_{c,p} - \delta_{bc}\hat{a}^\dagger_{d,p}\hat{a}_{a,p}\right) \right)
\]
\[
\langle \text{reorganizing by } \delta \text{'s} \rangle
\]
\[
= \sum_p \left( -\delta_{bc}\left(\hat{a}^\dagger_{a,p}\hat{a}_{d,p} - \hat{a}^\dagger_{d,p}\hat{a}_{a,p}\right) + \delta_{ac}\left(\hat{a}^\dagger_{b,p}\hat{a}_{d,p} - \hat{a}^\dagger_{d,p}\hat{a}_{b,p}\right) \\
+ \delta_{bd}\left(\hat{a}^\dagger_{a,p}\hat{a}_{c,p} - \hat{a}^\dagger_{c,p}\hat{a}_{a,p}\right) + \delta_{ad}\left(\hat{a}^\dagger_{b,p}\hat{a}_{c,p} - \hat{a}^\dagger_{c,p}\hat{a}_{b,p}\right) \right)
\]
\[
= -i\delta_{bc}\hat{Q}_{ad} + i\delta_{ac}\hat{Q}_{bd} + i\delta_{bd}\hat{Q}_{ac} - i\delta_{ad}\hat{Q}_{bc}.
\] (10)

Q.E.D.
Problem 2(g):
Note: the first two equations (12) for the annihilation operators \( \hat{a}_p \) and \( \hat{b}_p \) parallel the way real fields \( \Phi_1(x) \) and \( \Phi_2(x) \) combine into the complex field \( \Phi(x) \) and its hermitian conjugate \( \Phi^\dagger(x) \). The last two equations (12) for the creation operators follow from the first two equations by hermitian conjugation. Thanks to these definitions,

\[
\hat{\Phi}_p \equiv \frac{\hat{\Phi}_{a,p} + i\hat{\Phi}_{2,k}}{\sqrt{2}} = \frac{\hat{a}_{1,p} + \hat{a}_{1,-p}^\dagger + i\hat{a}_{2,p} + i\hat{a}_{2,-p}^\dagger}{\sqrt{2}\sqrt{2E_p}} \tag{S.52}
\]

and hence

\[
\hat{\Phi}(x) = \sum_p L^{-3/2}e^{ipx} \times \frac{\hat{a}_p + \hat{b}_p^\dagger}{\sqrt{2E_p}}. \tag{S.53}
\]

Note that this non-hermitian \( \hat{\Phi} \) fields contains the particle annihilation operators but the antiparticle creation operators! The remaining antiparticle annihilation operators and particle creation operators comprise the hermitian conjugate field \( \Phi^\dagger(x) \):

\[
\hat{\Phi}^\dagger(x) = \sum_p L^{-3/2}e^{-ipx} \times \frac{\hat{a}_p^\dagger + \hat{b}_p^\dagger}{\sqrt{2E_p}} = \sum_p L^{-3/2}e^{+ipx} \times \frac{\hat{b}_p + \hat{a}_p^\dagger}{\sqrt{2E_p}}. \tag{S.54}
\]

Eqs. (S.53) and (S.54) justify my definitions (12).

Problem 2(h):
Given the operator definitions (12), we have

\[
\hat{a}_p^\dagger\hat{a}_p - \hat{b}_p^\dagger\hat{b}_p = \frac{1}{2}(\hat{a}_{1,p}^\dagger - i\hat{a}_{2,p}^\dagger)(\hat{a}_{1,p} + i\hat{a}_{2,p}) - \frac{1}{2}(\hat{a}_{1,p}^\dagger + i\hat{a}_{2,p}^\dagger)(\hat{a}_{1,p} - i\hat{a}_{2,p}) \tag{S.55}
\]

= \ -i\hat{a}_{2,p}^\dagger\hat{a}_{1,p} + i\hat{a}_{1,p}^\dagger\hat{a}_{2,p}.
and therefore

\[ N_{\text{particles}} - N_{\text{antiparticles}} = \sum_P \left( \hat{a}^\dagger_P \hat{a}_P - \hat{b}^\dagger_P \hat{b}_P \right) \]

\[ = \sum_P \left( -i\hat{a}^\dagger_{2P} \hat{a}_{1P} + i\hat{a}^\dagger_{1P} \hat{a}_{2P} \right) \]

\[ = \hat{Q}_{21} = -\hat{Q}_{12}. \quad \text{(S.56)} \]

\[ Q.E.D. \]

Problem 3(a):
First, let’s show that the creation operators defined according to eq. (16) commute with each other. Pick any two such creation operators \( \hat{a}_\alpha \) and \( \hat{a}_\beta \), and pick any \( N \)-boson state \( |N, \psi\rangle \). Consider the \( (N+2) \)-boson wavefunction \( \psi'''(x_1, \ldots, x_{N+2}) \) of the state \( |N + 2, \psi'''\rangle = \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger |N, \psi\rangle \).

Applying eq. (16) twice, we immediately obtain

\[ \psi'''(x_1, \ldots, x_{N+2}) = \frac{1}{\sqrt{(N+1)(N+2)}} \sum_{i,j=1,\ldots,N+2}^{i \neq j} \phi_\alpha(x_i) \times \phi_\beta(x_j) \times \psi(x_1, \ldots, x_{N+2} \text{ except } x_i, x_j). \quad \text{(S.57)} \]

On the RHS of this formula, interchanging \( \alpha \leftrightarrow \beta \) is equivalent to interchanging the summation indices \( i \leftrightarrow j \) — which has no effect on the sum. Consequently, the states \( \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger |N, \psi\rangle \) and \( \hat{a}_\beta^\dagger \hat{a}_\alpha^\dagger |N, \psi\rangle \) have the same wavefunction (S.57), thus

\[ \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger |N, \psi\rangle = \hat{a}_\beta^\dagger \hat{a}_\alpha^\dagger |N, \psi\rangle. \quad \text{(S.58)} \]

Since this is true for any \( N \) and any totally-symmetric wave function \( \psi \), this means that the creation operators \( \hat{a}_\alpha^\dagger \) and \( \hat{a}_\beta^\dagger \) commute with each other.

Next, let’s pick any two annihilation operators \( \hat{a}_\alpha \) and \( \hat{a}_\beta \) defined according to eq. (17) and show that they commute with each other. Again, let \( |N, \psi\rangle \) be an arbitrary \( N \)-boson state . For \( N < 2 \) we have

\[ \hat{a}_\alpha \hat{a}_\beta |N, \psi\rangle = 0 = \hat{a}_\beta \hat{a}_\alpha^\dagger |N, \psi\rangle, \quad \text{(S.59)} \]

so let’s focus on the non-trivial case of \( N \geq 2 \) and consider the \( (N - 2) \)-boson wavefunction
\( \psi^{'''\prime\prime\prime} \) of the state \(|N - 2, \psi^{'''\prime\prime\prime}\rangle = \hat{a}_\alpha \hat{a}_\beta |N, \psi \rangle \). Applying eq. (17) twice, we obtain

\[
\psi^{'''\prime\prime\prime}(x_1, \ldots, x_{N-2}) = \sqrt{N(N-1)} \int d^3x_N \int d^3x_{N-1} \phi^*_\alpha(x_N) \times \phi^*_\beta(x_{N-1}) \times \psi(x_1, \ldots, x_{N-2}, x_{N-1}, x_N).
\]

On the RHS of this formula, interchanging \( \alpha \leftrightarrow \beta \) is equivalent to interchanging the integrated-over positions of the \( N^{\text{th}} \) and the \((N-1)^{\text{th}} \) boson in the original state \(|N, \psi \rangle \). Thanks to bosonic symmetry of the wave-function \( \psi \), this interchange has no effect, thus

\[
\hat{a}_\alpha \hat{a}_\beta |N, \psi \rangle = \hat{a}_\beta \hat{a}_\alpha |N, \psi \rangle.
\]

Therefore, when the annihilation operators defined according to eq. (17) act on the totally-symmetric wave functions of identical bosons, they commute with each other.

Finally, let’s pick a creation operator \( \hat{a}_\beta^\dagger \) and an annihilation operator \( \hat{a}_\alpha \), pick an arbitrary \( N \)-boson state \(|N, \psi \rangle \), and consider the difference between the states

\[
|N, \psi^5 \rangle = \hat{a}_\beta^\dagger \hat{a}_\alpha |N, \psi \rangle \quad \text{and} \quad |N, \psi^6 \rangle = \hat{a}_\alpha \hat{a}_\beta^\dagger |N, \psi \rangle.
\]

Suppose \( N > 0 \). Applying eq. (17) to the wave function \( \psi \) and then applying eq. (16) to the result, we obtain

\[
\psi^5(x_1, \ldots, x_N) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \phi_\beta(x_i) \times \psi^{''\prime}(x_1, \ldots, \hat{x}_i, \ldots, x_N) \\
= \sum_{i=1}^{N} \phi_\beta(x_i) \times \int d^3x_{N+1} \phi^*_\alpha(x_{N+1}) \times \psi(x_1, \ldots, \hat{x}_i, \ldots, x_N, x_{N+1}).
\]

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On the other hand, applying first eq. (16) and then eq. (17), we arrive at

\[ \psi^6(x_1, \ldots, x_N) = \sqrt{N+1} \int d^3x_{N+1} \phi_{\alpha}^*(x_{N+1}) \times \psi'(x_1, \ldots, x_N, x_{N+1}) \]

\[ = \int d^3x_{N+1} \phi_{\alpha}^*(x_{N+1}) \times \sum_{i=1}^{N+1} \phi_{\beta}(x_i) \times \psi(x_1, \ldots, x_i, \ldots, x_{N+1}) \]

\[ = \int d^3x_{N+1} \phi_{\alpha}^*(x_{N+1}) \times \left( \sum_{i=1}^{N} \phi_{\beta}(x_i) \times \psi(x_1, \ldots, x_i, \ldots, x_N, x_{N+1}) \right) \]

\[ + \psi(x_1, \ldots, x_N) \times \int d^3x_{N+1} \phi_{\alpha}^*(x_{N+1}) \times \phi_{\beta}(x_{N+1}) \]

\[ = \psi^5(x_1, \ldots, x_N) \quad \langle \text{compare to eq. (S.63)} \rangle \]

\[ + \psi(x_1, \ldots, x_N) \times \langle \phi_{\alpha} | \phi_{\beta} \rangle. \]  

(S.64)

Comparing eqs. (S.63) and (S.64), we see that

\[ \psi^6(x_1, \ldots, x_N) - \psi^5(x_1, \ldots, x_N) = \psi(x_1, \ldots, x_N) \times \langle \phi_{\alpha} | \phi_{\beta} \rangle = \psi(x_1, \ldots, x_N) \times \delta_{\alpha\beta}, \]  

(S.65)

where \( \langle \phi_{\alpha} | \phi_{\beta} \rangle = \delta_{\alpha\beta} \) by orthonormality of the 1-particle basis \{\phi_{\gamma}(x)\}_{\gamma}. In Dirac notations, eq. (S.65) amounts to

\[ \left( \hat{a}_{\alpha} \hat{a}_{\beta} - \hat{a}_{\beta} \hat{a}_{\alpha} \right) |N, \psi\rangle = |N, \psi\rangle \times \delta_{\alpha\beta}. \]  

(S.66)

Thus far, we have checked this formula for all bosonic states \(|N, \psi\rangle\) except for the vacuum \(|0\rangle\). To complete the proof, note that

\[ \hat{a}_{\alpha} |0\rangle = 0 \quad \Rightarrow \quad \hat{a}_{\alpha} \hat{a}_{\alpha} |0\rangle = 0, \]  

(S.67)

while

\[ \hat{a}_{\alpha} \hat{a}_{\beta} ^{\dagger} |0\rangle = \hat{a}_{\alpha} |1, \phi_{\beta} \rangle = \langle \phi_{\alpha} | \phi_{\beta} \rangle \times |0\rangle = \delta_{\alpha\beta} \times |0\rangle, \]  

(S.68)
hence

\[ (\hat{a}_\alpha \hat{a}^\dagger_\beta - \hat{a}^\dagger_\beta \hat{a}_\alpha) |0\rangle = \delta_{\alpha\beta} \times |0\rangle. \]  

(S.69)

Altogether, eqs. (S.66) and (S.69) verify that

\[ [\hat{a}_\alpha, \hat{a}^\dagger_\beta] |\Psi\rangle = \delta_{\alpha\beta} \Psi \]  

(S.70)

for any state \( \Psi \) in the bosonic Fock space, hence the operators \( \hat{a}_\alpha \) and \( \hat{a}^\dagger_\beta \) defined according to eqs. (16) and (17) indeed obey the commutation relation \( [\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta} \). \( Q.E.D. \)

Problem 3(b):

In wave-function terms,

\[
\langle N - 1, \tilde{\psi} | \hat{a}_\alpha | N, \psi \rangle = \int d^3 x_1 \cdots \int d^3 x_{N-1} \tilde{\psi}^* (x_1, \ldots, x_{N-1}) \times \psi'' (x_1, \ldots, x_{N-1}) \\
= \int d^3 x_1 \cdots \int d^3 x_{N-1} \tilde{\psi}^* (x_1, \ldots, x_{N-1}) \times \\
\times \sqrt{N} \int d^3 x_N \phi^*_\alpha \times \psi (x_1, \ldots, x_N) \\
= \sqrt{N} \int d^3 x_1 \cdots \int d^3 x_N \tilde{\psi}^* (x_1, \ldots, x_{N-1}) \times \phi^*_\alpha (x_N) \times \psi (x_1, \ldots, x_N). 
\]  

(S.71)

At the same time,

\[
\langle N, \psi | \hat{a}^\dagger_\alpha | N - 1, \tilde{\psi} \rangle = \int d^3 x_1 \cdots \int d^3 x_N \psi^* (x_1, \ldots, x_N) \times \tilde{\psi}' (x_1, \ldots, x_N) \\
= \int d^3 x_1 \cdots \int d^3 x_N \psi^* (x_1, \ldots, x_N) \times \\
\times \frac{1}{\sqrt{N}} \sum_{i=1}^N \phi_\alpha (x_i) \times \tilde{\psi} (x_1, \ldots, x_i, \ldots, x_N) \\
= \frac{1}{\sqrt{N}} \sum_{i=1}^N \int d^3 x_1 \cdots \int d^3 x_N \psi^* (x_1, \ldots, x_N) \times \\
\times \phi_\alpha (x_i) \times \tilde{\psi} (x_1, \ldots, x_i, \ldots, x_N). 
\]  

(S.72)

By bosonic symmetry of the wavefunctions \( \psi \) and \( \tilde{\psi} \), all terms in the sum on the RHS are equal to each other. So, we may replace the summation with a single term — say, for \( i = N \) — and
multiply by $N$, thus

$$
\langle N, \psi | \hat{a}_\alpha^\dagger | N - 1, \tilde{\psi} \rangle = \frac{N}{\sqrt{N}} \times \int d^3x_1 \cdots \int d^3x_N \psi^*(x_1, \ldots, x_N) \times \phi_\alpha(x_N) \times \tilde{\psi}(x_1, \ldots, x_{N-1}).
$$

(S.73)

By inspection, the RHS of eqs. (S.71) and (S.73) are complex conjugates of each other, hence

$$
\langle N - 1, \tilde{\psi} | \hat{a}_\alpha | N, \psi \rangle = \langle N, \psi | \hat{a}_\alpha^\dagger | N - 1, \tilde{\psi} \rangle^*.
$$

(18)

Q.E.D.

Problem 3(c):

First, a note on the $1/\sqrt{T}$ in eq. (14). We need this factor to properly normalize the multi-boson states in which some bosons may be in the same 1-particle mode. For example, for the two particle states,

$$
|\alpha, \beta\rangle = \hat{a}_\beta^\dagger \hat{a}_\alpha^\dagger |0\rangle \quad \text{when } \alpha \neq \beta, \quad \text{but } |\alpha, \alpha\rangle = \frac{1}{\sqrt{2}} \hat{a}_\alpha^\dagger \hat{a}_\alpha^\dagger |0\rangle.
$$

(S.74)

In terms of the occupation numbers, the properly normalized states are

$$
|\{n_\alpha\}_\alpha\rangle = \bigotimes_\alpha (|n_\alpha\rangle = \frac{(\hat{a}_\alpha^\dagger)^{n_\alpha}}{\sqrt{n_\alpha!}} |0\rangle)_{\text{mode } \alpha} = \left( \prod_\alpha \frac{(\hat{a}_\alpha^\dagger)^{n_\alpha}}{\sqrt{n_\alpha!}} \right) |\text{vacuum}\rangle.
$$

(S.75)

hence eq. (14) in terms of the list.

Now let’s work out the wave functions of the states (14) by successively applying eq. (16).

1. For $N = 1$, states $|\alpha\rangle = \hat{a}_\alpha^\dagger |0\rangle$ have wave functions $\phi_\alpha(x)$.

2. For $N = 2$, states $\sqrt{T} |\alpha, \beta\rangle = \hat{a}_\beta^\dagger \hat{a}_\alpha^\dagger |0\rangle$ have wavefunctions

$$
\sqrt{T} \times \phi_{\alpha,\beta}(x_1, x_2) = \frac{1}{\sqrt{2}} \left( \phi_\beta(x_1) \phi_\alpha(x_2) + \phi_\beta(x_2) \phi_\alpha(x_1) \right).
$$

(S.76)
3. For \( N = 3 \), states \( \sqrt{T} |\alpha, \beta, \gamma\rangle = \hat{a}_\gamma \hat{a}_\beta \hat{a}_\alpha |0\rangle \) have

\[
\sqrt{T} \times \phi_{\alpha\beta\gamma}(x_1, x_2, x_3) = \frac{1}{\sqrt{3}} \left( \phi_\gamma(x_1) \times \frac{1}{\sqrt{2}} (\phi_\beta(x_2)\phi_\alpha(x_3) - \phi_\beta(x_3)\phi_\alpha(x_2)) \right) \\
+ \phi_\gamma(x_2) \times \frac{1}{\sqrt{2}} (\phi_\beta(x_1)\phi_\alpha(x_3) - \phi_\beta(x_3)\phi_\alpha(x_1)) \\
+ \phi_\gamma(x_3) \times \frac{1}{\sqrt{2}} (\phi_\beta(x_1)\phi_\alpha(x_2) - \phi_\beta(x_2)\phi_\alpha(x_1))
\]

\[
= \frac{1}{\sqrt{3}} \sum_{(x_1, x_2, x_3)} \phi_\gamma(\bar{x}_1)\phi_\beta(\bar{x}_2)\phi_\alpha(\bar{x}_3)
\]

\[
= \frac{1}{\sqrt{3}} \sum_{(\bar{\alpha}, \bar{\beta}, \bar{\gamma})} \phi_{\bar{\alpha}}(x_1)\phi_{\bar{\beta}}(x_2)\phi_{\bar{\gamma}}(x_3).
\]

Extrapolating from eq. (S.77), the \( N \)-particle state \( \sqrt{T} |\alpha, \ldots, \omega\rangle = \hat{a}_\omega \cdots \hat{a}_\alpha |0\rangle \), has the totally-symmetrized wave function

\[
\sqrt{T} \times \phi_{\alpha \ldots \omega}(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \sum_{(\bar{\alpha}, \ldots, \bar{\omega})} \phi_{\bar{\alpha}}(x_1) \times \cdots \times \phi_{\bar{\omega}}(x_N).
\]

Dividing both sides of this formula by the \( \sqrt{T} \) factor, we immediately arrive at the second line of eq. (15).

Finally, the top line of eq. (15) obtains from the bottom line by adding up coincident terms. Indeed, if some one-particle states appear multiple times in the list \( (\alpha, \ldots, \omega) \), then permuting coincident entries of this list has no effect. Altogether, there \( T \) such trivial permutations. By group theory, this means that out of \( N! \) possible permutations of the list, there are only \( N!/T \) distinct permutations. But for each distinct permutations, there are \( T \) coincident terms in the sum on the bottom line of eq. (15). Adding them up gives us the top line of eq. (15).
Problem 3(d):
Let $A_{\alpha \beta} = \langle \alpha | \hat{A}_1 | \beta \rangle$. Since states $|\alpha\rangle$ make a complete basis of the 1-particle Hilbert space, for any 1-particle states $\langle \tilde{\psi} |$ and $|\psi\rangle$

$$
\langle \tilde{\psi} | \hat{A}_1 | \psi \rangle = \sum_{\alpha, \beta} A_{\alpha \beta} \langle \tilde{\psi} | \alpha \rangle \langle \beta | \psi \rangle = \sum_{\alpha, \beta} A_{\alpha \beta} \times \int d^3 \tilde{x} \tilde{\psi}^*(\tilde{x}) \phi_{\alpha}(\tilde{x}) \times \int d^3 x \phi_{\beta}^*(x) \psi(x). \quad (S.79)
$$

This is undergraduate-level QM.

In the $N$-particle Hilbert space we have a similar formula for the matrix elements of the $\hat{A}_1$ acting on particle $#i$, the only modification being integrals over the coordinates of the other particles,

$$
\langle N, \tilde{\psi} | \hat{A}_1(i^{th}) | N, \psi \rangle = \int \cdots \int d^3 x_1 \cdots d^3 x_i \cdots d^3 x_N \sum_{\alpha, \beta} A_{\alpha \beta} \times \left( \int d^3 x_i \tilde{\psi}^*(x_i, \ldots, \tilde{x}_i, \ldots, x_N) \phi_{\alpha}(\tilde{x}_i) \right)
\times \left( \int d^3 x_i \phi_{\beta}^*(x_i) \psi(x_1, \ldots, x_i, \ldots, x_N) \right)
\sum_{\alpha, \beta} A_{\alpha \beta} \times \int \cdots \int d^3 x_1 \cdots d^3 x_i \cdots d^3 x_N \tilde{\psi}^*(x_1, \ldots, \tilde{x}_i, \ldots, x_N) \times \phi_{\alpha}(\tilde{x}_i)
\times \phi_{\beta}^*(x_i) \times \psi(x_1, \ldots, x_i, \ldots, x_N). \quad (S.80)
$$

For symmetric wave-functions of identical bosons, we get the same matrix element regardless of which particle $#i$ we are acting on with the operator $\hat{A}_1$, hence for the net $A$ operator (5.9),

$$
\langle N, \tilde{\psi} | \hat{A}_1^{(1)}_{\text{net}} | N, \psi \rangle = N \times \sum_{\alpha, \beta} A_{\alpha \beta} \times \int \cdots \int d^3 x_1 \cdots d^3 x_{N-1} d^3 x_N d^3 \tilde{x}_N
\tilde{\psi}^*(x_1, \ldots, x_{N-1}, \tilde{x}_N) \times \phi_{\alpha}(\tilde{x}_N)
\times \phi_{\beta}^*(x_N) \times \psi(x_1, \ldots, x_{N-1}, x_N). \quad (S.81)
$$

Now consider matrix elements of the Fock-space operator (20). In light of eq. (17), the state $|N - 1, \psi''\rangle = \hat{a}_\beta |N, \psi\rangle$ has wave-function

$$
\psi''(x_1, \ldots, x_{N-1}) = \sqrt{N} \int d^3 x_N \phi_{\beta}^*(x_N) \times \psi(x_1, \ldots, x_{N-1}, x_N). \quad (S.82)
$$
Likewise, the state $|N - 1, \tilde{\psi}'\rangle = \hat{a}_\alpha |N, \tilde{\psi}\rangle$ has wave-function

$$
\tilde{\psi}'(x_1, \ldots, x_{N-1}) = \sqrt{N} \int d^3 \tilde{x}_N \phi_\alpha^*(\tilde{x}_N) \times \tilde{\psi}(x_1, \ldots, x_{N-1}, \tilde{x}_N).
$$

Consequently,

$$
\langle N, \tilde{\psi}| \hat{a}_\alpha^\dagger \hat{a}_\beta |N, \psi\rangle = \langle N - 1, \tilde{\psi}'| |N - 1, \psi''\rangle
= \int \cdots \int d^3 x_1 \cdots x_{N-1} \tilde{\psi}''(x_1, \ldots, x_{N-1}) \times \tilde{\psi}''(x_1, \ldots, x_{N-1})
= \int \cdots \int d^3 x_1 \cdots x_{N-1} \sqrt{N} \int d^3 \tilde{x}_N \phi_\alpha(\tilde{x}_N) \times \tilde{\psi}^*(x_1, \ldots, x_{N-1}, \tilde{x}_N) \times
\times \sqrt{N} \int d^3 \tilde{x}_N \phi_\beta^*(\tilde{x}_N) \times \psi(x_1, \ldots, x_{N-1}, x_N).
$$

Comparing this formula to the integrals in eq. (S.81), we see that

$$
\langle N, \tilde{\psi}| \hat{A}_{\text{net}}^{(1)} |N, \psi\rangle = \sum_{\alpha, \beta} A_{\alpha \beta} \times \langle N, \tilde{\psi}| \hat{a}_\alpha^\dagger \hat{a}_\beta |N, \psi\rangle = \langle N, \tilde{\psi}| \hat{A}_{\text{net}}^{(2)} |N, \psi\rangle.
$$

Q.E.D.

Problem 3(e):
This part follows from the commutator (S.3) in problem 1(a). Indeed, given

$$
\hat{A}_{\text{tot}}^{(2)} = \sum_{\alpha, \beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta
$$

and

$$
\hat{B}_{\text{tot}}^{(2)} = \sum_{\gamma, \delta} \langle \gamma | \hat{B}_1 | \delta \rangle \hat{a}_\gamma^\dagger \hat{a}_\delta,
$$
we immediately have

\[
\left[ \hat{A}_{\text{tot}}^{(2)}, \hat{B}_{\text{tot}}^{(2)} \right] = \sum_{\alpha,\beta,\gamma,\delta} \langle \alpha| \hat{A}_1 | \beta \rangle \langle \gamma| \hat{B}_1 | \delta \rangle \left[ \hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta \right]
\]

\[
\langle \text{using \ (S.3)} \rangle
\]

\[
= \sum_{\alpha,\beta,\gamma,\delta} \langle \alpha| \hat{A}_1 | \beta \rangle \langle \gamma| \hat{B}_1 | \delta \rangle \left( \delta_{\beta,\gamma} \delta_{\alpha,\delta} - \delta_{\alpha,\delta} \delta_{\beta,\gamma} \right)
\]

\[
= \sum_{\alpha,\beta,\gamma,\delta} \hat{a}^\dagger_\alpha \hat{a}_\delta \times \sum_{\beta=\gamma} \langle \alpha| \hat{A}_1 | \gamma \rangle \langle \gamma| \hat{B}_1 | \delta \rangle - \sum_{\beta,\gamma} \hat{a}^\dagger_\gamma \hat{a}_\beta \times \sum_{\alpha=\delta} \langle \gamma| \hat{B}_1 | \alpha \rangle \langle \alpha| \hat{A}_1 | \beta \rangle
\]

\[
\langle \text{renaming summation indices} \rangle
\]

\[
= \sum_{\alpha,\beta} \hat{a}^\dagger_\alpha \hat{a}_\beta \times \left( \langle \alpha| \hat{A}_1 \hat{B}_1 | \beta \rangle - \langle \alpha| \hat{B}_1 \hat{A}_1 | \beta \rangle \right)
\]

\[
= \sum_{\alpha,\beta} \hat{a}^\dagger_\alpha \hat{a}_\beta \times \langle \alpha| \left[ \hat{A}_1, \hat{B}_1 \right] = \hat{C}_1 \rangle | \beta \rangle \equiv \hat{C}^{(2)}_{\text{tot}}.
\]

(S.88)

**Problem 3(f):**

This works similarly to part (d), except for more integrals 😊. Let

\[
B_{\alpha\beta,\gamma\delta} = \left( \langle \alpha| \otimes \langle \beta| \right) \hat{B}_2 \left( | \gamma \rangle \otimes | \delta \rangle \right)
\]

(S.89)

be matrix elements of a two-body operator \( \hat{B}_2 \) between *un-symmetrized* two-particle states. Then for generic two-particle states \( |\tilde{\psi}\rangle \) and \( |\psi\rangle \) — symmetric or not — we have

\[
\langle \tilde{\psi}| \hat{B}_2 |\psi\rangle = \sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta,\gamma\delta} \times \langle \tilde{\psi}| (|\alpha\rangle \otimes |\beta\rangle) \times (\langle \gamma| \otimes \langle \delta|) |\psi\rangle
\]

\[
= \sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta,\gamma\delta} \times \int d^3\bar{x}_1 d^3\bar{x}_2 \tilde{\psi}^* (\bar{x}_1, \bar{x}_2) \phi_\alpha (\bar{x}_1) \phi_\beta (\bar{x}_2)
\]

\[
\times \int d^3x_1 d^3x_2 \phi^*_\gamma (x_1) \phi^*_\delta (x_2) \psi (x_1, x_2).
\]

(S.90)

Similarly, in the Hilbert space of \( N > 2 \) particles — identical bosons or not — the operator \( \hat{B}_2 \)
acting on particles \#i and \#j has matrix elements

\[
\langle N, \tilde{\psi} | \hat{B}_2(i^{\text{th}}, j^{\text{th}}) | N, \psi \rangle = \sum_{\alpha, \beta, \gamma, \delta} B_{\alpha\beta\gamma\delta} \times \int \cdots \int d^3x_1 \cdots d^3x_i \cdots d^3x_{N-1} \cdots d^3x_N \int d^3x_i d^3x_j \tilde{\psi}^*(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) \phi_\alpha(\tilde{x}_i) \phi_\beta(\tilde{x}_j) \times \int d^3x_i d^3x_j \phi_\gamma^*(x_i) \phi_\delta^*(x_j) \psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N)
\]

For identical bosons — and hence totally symmetric wave-functions \(\psi\) and \(\tilde{\psi}\) — such matrix elements do not depend on the choice of particles on which \(\hat{B}_2\) acts, so we may just as well let \(i = N - 1\) and \(j = N\). Consequently, the net \(\hat{B}\) operator (21) has matrix elements

\[
\langle N, \tilde{\psi} | \hat{B}_{\text{net}}^{(1)} | N, \psi \rangle = \frac{N(N-1)}{2} \times \langle N, \tilde{\psi} | \hat{B}_2(N-1, N) | N, \psi \rangle = \frac{N(N-1)}{2} \sum_{\alpha, \beta, \gamma, \delta} B_{\alpha\beta\gamma\delta} \times I_{\alpha\beta\gamma\delta}
\]

where

\[
I_{\alpha\beta\gamma\delta} = \int \cdots \int d^3x_1 \cdots d^3x_{N-2} \int d^3x_{N-1} d^3x_N \tilde{\psi}^*(x_1, \ldots, x_{N-2}, x_{N-1}, x_N) \phi_\alpha(\tilde{x}_{N-1}) \phi_\beta(\tilde{x}_N) \times \int d^3x_{N-1} d^3x_N \phi_\gamma^*(x_{N-1}) \phi_\delta^*(x_N) \psi(x_1, \ldots, x_{N-2}, x_{N-1}, x_N)
\]

Now let’s compare this to the Fock space formalism. Applying eq. (17) twice, we find that the \((N - 2)\)–particle state

\[
| N - 2, \psi''' \rangle = \hat{a}_\delta \hat{a}_\gamma | N, \psi \rangle
\]

has wave function

\[
\psi'''(x_1, \ldots, x_{N-2}) = \sqrt{N(N-1)} \int d^3x_{N-1} d^3x_N \phi_\gamma^*(x_{N-1}) \phi_\delta^*(x_N) \times \psi(x_1, \ldots, x_{N-2}, x_{N-1}, x_N).
\]

Likewise, the \((N - 2)\)–particle state

\[
| N - 2, \tilde{\psi}''' \rangle = \hat{a}_\delta \hat{a}_\alpha | N, \tilde{\psi} \rangle
\]
has wave function

\[
\psi'''(x_1, \ldots, x_{N-2}) = \sqrt{N(N-1)} \int \cdots \int d^3x_{N-1} d^3x_N \phi^*_\alpha(\bar{x}_{N-1}) \phi^*_\alpha(\bar{x}_N) \\
\times \tilde{\psi}(x_1, \ldots, x_{N-2}, \bar{x}_{N-1}, \bar{x}_N).
\]  

(S.96)

Taking Dirac product of these two states, we obtain

\[
\langle N, \tilde{\psi} | \hat{A}^\dagger_\alpha \hat{A}^\dagger_\beta \hat{A}_\delta \hat{A}_\gamma | N, \psi \rangle = \langle N - 2, \tilde{\psi}''' | N - 2, \psi''' \rangle = \int \cdots \int d^3x_1 \cdots d^3x_{N-2} \psi'''(x_1, \ldots, x_{N-2}) \times \psi'''(x_1, \ldots, x_{N-2}) \\
= N(N-1) \times I_{\alpha,\beta,\gamma,\delta}
\]

(S.97)

where \( I_{\alpha,\beta,\gamma,\delta} \) is exactly the same mega-integral as in eq. (S.92). Therefore,

\[
\langle N, \tilde{\psi} | \hat{B}^{(1)}_{\text{net}} | N, \psi \rangle = \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \langle N, \tilde{\psi} | \hat{A}^\dagger_\alpha \hat{A}^\dagger_\beta \hat{A}_\delta \hat{A}_\gamma | N, \psi \rangle = \langle N, \tilde{\psi} | \hat{B}^{(2)}_{\text{net}} | N, \psi \rangle
\]

(S.98)

where the second equality follows from eq. (22). \( Q.E.D. \)

**Problem 3(g):**

In the Fock space,

\[
\hat{A}^{(2)}_{\text{tot}} = \sum_{\mu,\nu} \langle \mu | \hat{A}_1 | \nu \rangle \hat{a}^\dagger_\mu \hat{a}_\nu
\]

(20)

and

\[
\hat{B}^{(2)}_{\text{tot}} = \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta,
\]

(22)

where \( \langle \alpha \otimes \beta \rangle \) is a short-hand for the un-symmetrized two-particle wave function \( (|\alpha\rangle \otimes |\beta\rangle) \)
and likewise $|\gamma \otimes \delta\rangle = (|\gamma \rangle \otimes |\delta\rangle)$. Therefore,

$$
[\hat{A}_{\text{tot}}^{(2)}, \hat{B}_{\text{tot}}^{(2)}] = \frac{1}{2} \sum_{\mu, \nu, \alpha, \beta, \gamma, \delta} \langle \mu | \hat{A}_1 | \nu \rangle \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle \left[ \hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger \hat{a}_\beta \right]
$$

\[\text{using eq. (S.4)}\]

$$
= \frac{1}{2} \sum_{\mu, \nu, \alpha, \beta, \gamma, \delta} \hat{a}_\mu^\dagger \hat{a}_\beta \hat{a}_\gamma \hat{a}_\delta \times \sum_{\nu} \langle \mu | \hat{A}_1 | \nu \rangle \langle \nu \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle
$$

$$
+ \frac{1}{2} \sum_{\alpha, \mu, \gamma, \delta} \hat{a}_\alpha^\dagger \hat{a}_\mu \hat{a}_\gamma \hat{a}_\delta \times \sum_{\nu} \langle \mu | \hat{A}_1 | \nu \rangle \langle \alpha \otimes \nu | \hat{B}_2 | \gamma \otimes \delta \rangle
$$

$$
- \frac{1}{2} \sum_{\alpha, \beta, \nu, \delta} \hat{a}_\alpha^\dagger \hat{a}_\beta \hat{a}_\nu \hat{a}_\delta \times \sum_{\mu} \langle \alpha \otimes \beta | \hat{B}_2 | \mu \otimes \delta \rangle \langle \mu | \hat{A}_1 | \nu \rangle
$$

$$
- \frac{1}{2} \sum_{\alpha, \beta, \gamma, \nu} \hat{a}_\alpha^\dagger \hat{a}_\beta \hat{a}_\gamma \hat{a}_\nu \times \sum_{\mu} \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \mu \rangle \langle \mu | \hat{A}_1 | \nu \rangle
$$

\[\text{using eq. (S.4)}\]

$$
= \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \hat{a}_\alpha^\dagger \hat{a}_\beta \hat{a}_\gamma \hat{a}_\delta \times C_{\alpha, \beta, \gamma, \delta},
$$

where

$$
C_{\alpha, \beta, \gamma, \delta} = \sum_\lambda \langle \alpha | \hat{A}_1 | \lambda \rangle \langle \lambda \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle + \sum_\lambda \langle \beta | \hat{A}_1 | \lambda \rangle \langle \alpha \otimes \lambda | \hat{B}_2 | \gamma \otimes \delta \rangle
$$

$$
- \sum_\lambda \langle \alpha \otimes \beta | \hat{B}_2 | \lambda \otimes \delta \rangle \langle \lambda | \hat{A}_1 | \gamma \rangle - \sum_\lambda \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \lambda \rangle \langle \lambda | \hat{A}_1 | \delta \rangle
$$

$$
= \langle \alpha \otimes \beta | \left( \hat{A}_1^{(1st)} \hat{B}_2 + \hat{A}_1^{(2nd)} \hat{B}_2 - \hat{B}_2 \hat{A}_1^{(1st)} - \hat{B}_2 \hat{A}_1^{(2nd)} \right) | \gamma \otimes \delta \rangle
$$

$$
= \langle \alpha \otimes \beta | \left( \hat{A}_1^{(1st)} + \hat{A}_1^{(2nd)} \right) | \gamma \otimes \delta \rangle \equiv \langle \alpha \otimes \beta | \hat{C}_2 | \gamma \otimes \delta \rangle.
$$

Consequently, $[\hat{A}_{\text{tot}}^{(2)}, \hat{B}_{\text{tot}}^{(2)}] = \hat{C}_{\text{tot}}^{(2)}. \quad Q.E.D.$