Problem 1(a):
The Hamiltonian (2) of a free relativistic particle — and hence the evolution operator \( \exp(-it\hat{H}) \) — are functions of the momentum operator \( \hat{p} \), so they diagonalize in the momentum basis. In the non-relativistic normalization of \( |k\rangle \) states,

\[
\hat{H} = \int \frac{d^3k}{(2\pi)^3} |k\rangle \omega(k) \langle k|,
\]

\[
\exp(-i\hat{H}t) = \int \frac{d^3k}{(2\pi)^3} |k\rangle \exp(-it\omega(k)) \langle k|,
\]

where \( t = y^0 - x^0 \) and \( \omega(k) = \sqrt{k^2 + M^2} \). In the same non-relativistic normalization \( \langle x|k\rangle = \exp(ik \cdot k) \), therefore

\[
U(x \to y) \equiv \langle y|\exp(-i\hat{H}t)|x\rangle = \int \frac{d^3k}{(2\pi)^3} \exp(ik \cdot (y - x) - it\omega(k)).
\]

In terms of the \( D \) function defined in class,

\[
D(y - x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega(k)} \times \exp(ik \cdot (y - x) - it\omega(k)),
\]

\[
2i \frac{\partial}{\partial y^0} D(y - x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega(k)} \times 2\omega(k) \exp(ik \cdot (y - x) - it\omega(k))
\]

\[
= U(x \to y), \quad \langle\rangle \text{ see eq. (S.2).} \]

This formula and eq. (S.2) verify eq. (4).

Now let’s simplify the 3D integral on the RHS of (S.2) by integrating over the directions of the \( k \) vector. In spherical coordinates \( (k, \theta, \phi) \) where \( k = |k| \) and \( \theta \) is the angle between \( k \) and \( y - x \),

\[
d^3k = dk k^2 d\cos \theta d\phi, \quad k \cdot (y - x) = rk \cos \theta,
\]

hence

\[
\int \int d\cos \theta d\phi e^{ik(x-y)} = 2\pi \int_{-1}^{+1} d\cos \theta e^{ikr \cos \theta} = \frac{2\pi}{irk} \left(e^{+irk} - e^{-irk}\right). \]
Consequently, for any symmetric function \( f(k) = f(-k) \) we have

\[
\int \frac{d^3k}{(2\pi)^3} e^{ik \cdot (y-x)} \times f(|k|) = \frac{1}{4\pi^2} \int_0^\infty dk k^2 \int_0^\infty \frac{1}{irk} \left( e^{+irk} - e^{-irk} \right) \times f(k) \\
= \frac{1}{4\pi^2 i r} \int_0^\infty dk k \left( e^{+irk} \times f(k) - e^{-irk} \times (f(k) = f(-k)) \right) \\
= \frac{1}{4\pi^2 i r} \int_{-\infty}^{+\infty} dk k e^{+irk} \times f(k).
\]

(S.6)

In particular, for the \( f(k) = e^{-it\omega(k)} = f(-k) \) we have

\[
U(x \to y) = \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot (y-x)} \times e^{-it\omega(k)} = \frac{1}{4\pi^2 i r} \int_{-\infty}^{+\infty} dk k e^{+irk} \times e^{-it\omega(k)}
\]

(S.7)

in accordance with eq. (5).

Problem 1(b):

As explained in my notes on the saddle point method integrals of the form

\[
I = \int_{\Gamma} dz f(z) e^{Ag(z)}
\]

(S.8)

in the large \( A \) limit become

\[
I = e^{Ag(z_0)} \times \frac{\sqrt{\pi} \eta f(z_0)}{\sqrt{-\eta^2 A g''(z_0)}} \times \left( 1 + O(A^{-1}) \right).
\]

(S.9)

In general, \( f \) and \( g \) are complex analytic functions of a complex variable \( z \) which is integrated over some contour \( \Gamma \); quite often \( \Gamma \) is the real axis, but one should allow for its deformation in the complex plane. In eq. (S.9), \( z_0 \) is a saddle point of \( g(z) \) where the derivative \( g'(z_0) = 0 \); this saddle point does not have to lie on the original integration contour \( \Gamma \) — if it does not, we deform the contour \( \Gamma \to \Gamma' \) so that \( \Gamma' \) does go through the \( z_0 \). If several saddle points
are present near the contour $\Gamma$, the point with the largest $\text{Re} g$ dominates the integral. Finally, $\eta$ is the direction $dz$ of the $\Gamma'$ at the saddle point $z_0$; it should be chosen such that $\text{Re}(-\eta^2 g''(z_0)) \geq 0$, which assures that $\Gamma'$ crosses $z_0$ as a mountain path, from a valley to the lowest crossing point to another valley.

For the integral (5) at hand, we identify

$$A = t, \quad g(k) = \frac{r}{t}k - i\omega(k), \quad f(k) = \frac{k}{4\pi^2 i r}. \quad (S.10)$$

The saddle point in the $k$ plane follows from

$$\frac{dg}{dk} \equiv \frac{r}{t} - \frac{dw}{dk} \equiv \frac{r}{t} - \frac{i k}{\omega} = 0. \quad (S.11)$$

For $r < t$ this equation has a real solution, namely

$$k_0 = M \times \frac{r}{\sqrt{t^2 - r^2}}, \quad \omega(k_0) = M \times \frac{t}{\sqrt{t^2 - r^2}}. \quad (S.12)$$

At this point

$$A_g(k_0) = i rk_0 - it \omega(k_0) = i M \frac{r^2 - t^2}{\sqrt{t^2 - r^2}} = -i M \times \sqrt{t^2 - r^2},$$

$$f(k_0) = \frac{-i M}{4\pi^2} \frac{1}{\sqrt{t^2 - r^2}},$$

$$A_g''(k_0) \equiv \frac{-it M^2}{\omega^3(k_0)} = \frac{(t^2 - r^2)^{3/2}}{i M t^2}, \quad (S.13)$$

and the direction of the integration contour at $k_0$ should be in the fourth quadrant of the complex plane, $\text{arg}(\eta)$ between 0 and $-\pi/2$; the real-axis contour is marginally OK. Substituting all these data into eq. (S.9) gives us

$$\sqrt{\pi \eta} f(z_0) \sqrt{-\eta^2 A_g''(z_0)} = \frac{(-i M)^{3/2}}{4\pi^{3/2}} \times \frac{t}{(t^2 - r^2)^{5/4}} \quad (S.14)$$

and therefore

$$U(x \rightarrow y) = \exp \left( -i M \sqrt{t^2 - r^2} \right) \times \frac{(-i M)^{3/2}}{4\pi^{3/2}} \times \frac{t}{(t^2 - r^2)^{5/4}} \times \left( 1 + O \left( \frac{1}{M \sqrt{t^2 - r^2}} \right) \right) \quad (S.15)$$

in accordance with eq. (6).
Problem 1(c):
Again, we use the saddle point method to evaluate the integral (5). That is, we identify $A$, $g(k)$, and $f(k)$ according to eq. (S.10), and solve eq. (S.11) to find the saddle point. But this time, for $r > t$ the saddle point is imaginary

$$k_0 = \frac{iMr}{\sqrt{r^2 - t^2}}, \quad \omega(k_0) = \frac{iMt}{\sqrt{r^2 - t^2}},$$  \hspace{1cm} (S.16)

so the integration contour must be deformed away from the real axis. At the saddle point (S.16),

$$A g(k_0) = irk_0 - it\omega(k_0) = -M \times \sqrt{r^2 - t^2},$$

$$f(k_0) = \frac{M}{4\pi^2} \frac{1}{\sqrt{r^2 - t^2}},$$

$$A g''(k_0) \equiv \frac{-itM^2}{\omega^3(k_0)} = + \frac{(r^2 - t^2)^{3/2}}{Mt^2},$$  \hspace{1cm} (S.17)

all all real, and the deformed contour should cross $k_0$ in the imaginary direction, $\arg(\eta) = \frac{\pi}{2} \pm \frac{\pi}{4}$. Consequently, in eq. (S.9)

$$\frac{\sqrt{\pi} \eta f(z_0)}{\sqrt{-\eta^2} A g''(z_0)} = \frac{+iM^{3/2}}{4\pi^{3/2}} \times \frac{t}{(r^2 - t^2)^{5/4}}$$  \hspace{1cm} (S.18)

and therefore

$$U(x - y; t) = \exp \left( -M \sqrt{r^2 - t^2} \right) \times \frac{iM^{3/2}}{4\pi^{3/2}} \times \frac{t}{(r^2 - t^2)^{5/4}} \times \left( 1 + O \left( \frac{1}{M\sqrt{r^2 - t^2}} \right) \right)$$  \hspace{1cm} (S.19)

in accordance with eq. (7).

Note that the exponential factor here decays as one goes further outside the future light cone. In other words, the probability of a relativistic particle moving faster than light is exponentially small. But tiny as it is, this probability does not vanish, and this violates the relativistic causality.
Problem 2(a):
The relations $\hat{A}^\dagger_{k,\lambda} = -\hat{A}_{-k,\lambda}$, $\hat{E}^\dagger_{k,\lambda} = -\hat{E}_{-k,\lambda}$ follow from the hermiticity of quantum fields $\hat{A}(x)$ and $\hat{E}(x)$, and also and from the third eq. (13) for the polarization vectors $e_\lambda(\pm k)$:

$$\hat{A}^\dagger_{k,\lambda} = \int d^3 x e^{+ikx}e_\lambda(k) \cdot \hat{A}(x) = \int d^3 x e^{-i(-k)x}(-e^*_\lambda(-k)) \cdot \hat{A} = -\hat{A}_{-k,\lambda}, \quad (S.20)$$

and likewise

$$\hat{E}^\dagger_{k,\lambda} = \int d^3 x e^{+ikx}e_\lambda(k) \cdot \hat{E}(x) = \int d^3 x e^{-i(-k)x}(-e^*_\lambda(-k)) \cdot \hat{E} = -\hat{E}_{-k,\lambda}. \quad (S.21)$$

The equal-time commutation relations follow from eqs. (8): Obviously,

$$[\hat{A}_{k,\lambda}, \hat{A}_{k',\lambda'}] = 0, \quad [\hat{E}_{k,\lambda}, \hat{E}_{k',\lambda'}] = 0. \quad (S.22)$$

Less obviously,

$$[\hat{A}_{k,\lambda}, \hat{E}^\dagger_{k',\lambda'}] = \int d^3 x \int d^3 y e^{-ikx} \left(e^*_\lambda(k)^i \times e^{+ik'y}(e_{\lambda'}(k'))^j \times [\hat{A}^i(x), \hat{E}^j(y)]\right)$$

$$\quad = \int d^3 x \int d^3 y e^{-ikx} \left(e^*_\lambda(k)^i \times e^{+ik'y}(e_{\lambda'}(k'))^j \times (-i)\delta^{(3)}(x - y)\delta_{ij}\right)$$

$$\quad = -i \int d^3 x e^{-i(k-k')x} \times (e^*_\lambda(k) \cdot e_{\lambda'}(k'))$$

$$\quad = -i(2\pi)^3 \delta^{(3)}(k - k') \times (e^*_\lambda(k) \cdot e_{\lambda'}(k))$$

$$\quad = -i(2\pi)^3 \delta^{(3)}(k - k') \times \delta_{\lambda,\lambda'}, \quad (S.23)$$

or equivalently,

$$[\hat{A}_{k,\lambda}, \hat{E}_{k',\lambda'}] = +i(2\pi)^3 \delta^{(3)}(k + k') \times \delta_{\lambda,\lambda'}. \quad (S.24)$$

5
Problem 2(b):
There are four terms in the Hamiltonian density (9), so let us consider them one by one.
Combining Fourier transform with decomposition into polarization modes, it is easy to see that in light of eq. (11),

\[ \int d^3x \hat{E}^2(x) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \hat{E}^\dagger_{k,\lambda} \hat{E}_{k,\lambda} \]  
(25)

and likewise

\[ \int d^3x \hat{A}^2(x) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \hat{A}^\dagger_{k,\lambda} \hat{A}_{k,\lambda}. \]  
(26)

Furthermore, using eq. (12) we obtain

\[ \nabla \times \hat{A}(x) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda e^{ikx} \left( ik \times e_\lambda(k) = \lambda |k|e_\lambda(k) \right) \hat{A}_{k,\lambda} \]  
(27)

and hence

\[ \int d^3x \left( \nabla \times \hat{A}(x) \right)^2 = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \lambda^2 k^2 \hat{A}^\dagger_{k,\lambda} \hat{A}_{k,\lambda}. \]  
(28)

Finally, the first eq. (13) gives us

\[ \nabla \cdot \hat{E}(x) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda e^{ikx} \left( ik \cdot e_\lambda(k) = i |k| \delta_{\lambda,0} \right) \hat{E}_{k,\lambda} \]  
(29)

and hence

\[ \int d^3x \left( \nabla \cdot \hat{E}(x) \right)^2 = \int \frac{d^3k}{(2\pi)^3} k^2 \hat{E}^\dagger_{k,0} \hat{E}_{k,0}. \]  
(30)

Altogether, the Hamiltonian (9) totals up to

\[ \hat{H} = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \left( \frac{1}{2} \left( 1 + \frac{k^2}{m^2} \delta_{\lambda,0} \right) \times \hat{E}^\dagger_{k,\lambda} \hat{E}_{k,\lambda} + \frac{m^2 + \lambda^2 k^2}{2} \times \hat{A}^\dagger_{k,\lambda} \hat{A}_{k,\lambda} \right). \]  
(31)

Taking a closer look at the coefficients here and comparing them to the \( \omega_k \) and \( C_{k,\lambda} \) in...
eq. (16.b–c), we see that

\[
1 + \frac{k^2}{m^2} \times \delta_{\lambda,0} = \begin{cases} 
1 & \text{for } \lambda = \pm 1 \\
\omega_k^2/m^2 & \text{for } \lambda = 0
\end{cases} = C_{k,\lambda}
\]

while

\[
m^2 + \lambda^2 k^2 = \begin{cases} 
\omega_k^2/m^2 & \text{for } \lambda = \pm 1 \\
\omega_k^2 & \text{for } \lambda = 0
\end{cases} = \frac{\omega_k^2}{C_{k,\lambda}},
\]

thus the Hamiltonian (S.31) amounts to

\[
\hat{H} = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \left( \frac{C_{k,\lambda}}{2} \times \hat{E}_{k,\lambda}^\dagger \hat{E}_{k,\lambda} + \frac{\omega_k^2}{2C_{k,\lambda}} \times \hat{A}_{k,\lambda}^\dagger \hat{A}_{k,\lambda} \right).
\]

Problem 2(c):
Given eqs. (S.22) and (S.24), we have

\[
[\hat{a}_{k,\lambda}, \hat{a}_{k',\lambda}^\dagger] = -i\omega_k \sqrt{\frac{C_{k',\lambda'}}{C_{k,\lambda}}} \left( [\hat{A}_{k,\lambda}, \hat{E}_{k',\lambda'}] = (+i)(2\pi)^3 \delta^{(3)}(k + k') \delta_{\lambda,\lambda'} \right)
\]

\[
+ -i\omega_{k'} \sqrt{\frac{C_{k,\lambda}}{C_{k',\lambda'}}} \left( [\hat{E}_{k,\lambda}, \hat{A}_{k',\lambda'}] = (-i)(2\pi)^3 \delta^{(3)}(k + k') \delta_{\lambda,\lambda'} \right)
\]

\[
= \omega_k \times (2\pi)^3 \delta^{(3)}(k + k') \delta_{\lambda,\lambda'} - \omega_{k'} \times (2\pi)^3 \delta^{(3)}(k + k') \delta_{\lambda,\lambda'} = 0.
\]

Likewise, \([\hat{a}_{k,\lambda}^\dagger, \hat{a}_{k',\lambda}^\dagger] = 0\). On the other hand,

\[
[\hat{a}_{k,\lambda}, \hat{a}_{k',\lambda}^\dagger] = +i\omega_k \sqrt{\frac{C_{k',\lambda'}}{C_{k,\lambda}}} \left( [\hat{A}_{k,\lambda}, \hat{E}_{k',\lambda'}^\dagger] = (-i)(2\pi)^3 \delta^{(3)}(k - k') \delta_{\lambda,\lambda'} \right)
\]

\[
+ -i\omega_{k'} \sqrt{\frac{C_{k,\lambda}}{C_{k',\lambda'}}} \left( [\hat{E}_{k,\lambda}, \hat{A}_{k',\lambda'}^\dagger] = (+i)(2\pi)^3 \delta^{(3)}(k - k') \delta_{\lambda,\lambda'} \right)
\]

\[
= \omega_k \times (2\pi)^3 \delta^{(3)}(k - k') \delta_{\lambda,\lambda'} + \omega_{k'}(2\pi)^3 \delta^{(3)}(k - k') \delta_{\lambda,\lambda'}
\]

\[
= 2\omega_k \times (2\pi)^3 \delta^{(3)}(k - k') \delta_{\lambda,\lambda'}.
\]

Q.E.D.
Problem 2(d):
Given the operators $\hat{a}_{k,\lambda}$ and $\hat{a}^\dagger_{k,\lambda}$ defined according to eqs. (17), we have

$$\hat{a}^\dagger_{k,\lambda} \hat{a}_{k,\lambda} = \frac{\omega_k^2}{C_{k,\lambda}} \hat{A}^\dagger_{k,\lambda} \hat{A}_{k,\lambda} + C_{k,\lambda} \hat{E}^\dagger_{k,\lambda} \hat{E}_{k,\lambda} + i \omega_k \hat{E}^\dagger_{k,\lambda} \hat{A}_{k,\lambda} - i \omega_k \hat{A}^\dagger_{k,\lambda} \hat{E}_{k,\lambda}. \quad \text{(S.35)}$$

Consequently,

$$\int \frac{d^3k}{(2\pi)^3} \frac{2\omega_k}{\omega_k} \sum_{\lambda} \omega_k \hat{a}^\dagger_{k,\lambda} \hat{a}_{k,\lambda} = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \left( \frac{C_{k,\lambda}}{2} \times \hat{E}^\dagger_{k,\lambda} \hat{E}_{k,\lambda} + \frac{\omega_k^2}{2C_{k,\lambda}} \times \hat{A}^\dagger_{k,\lambda} \hat{A}_{k,\lambda} \right)$$

$$+ \int \frac{d^3k}{(2\pi)^3} \frac{i\omega_k}{2} \sum_{\lambda} \left( \hat{E}^\dagger_{k,\lambda} \hat{A}_{k,\lambda} - \hat{A}^\dagger_{k,\lambda} \hat{E}_{k,\lambda} \right);$$

(S.36)

where the first integral on the RHS is the Hamiltonian operator (16.a). Therefore, to verify eq. (18) we need to show that the second integral

$$\Delta \hat{H} \overset{\text{def}}{=} \int \frac{d^3k}{(2\pi)^3} \frac{i\omega_k}{2} \sum_{\lambda} \left( \hat{E}^\dagger_{k,\lambda} \hat{A}_{k,\lambda} - \hat{A}^\dagger_{k,\lambda} \hat{E}_{k,\lambda} \right)$$

(S.37)

is zero or a c-number constant.

The trick here is to separate the two terms under the integral, and then in the second term only apply eqs. (S.20) and (S.21) to the operators and change the integration variable $k \to -k$:

$$\int \frac{d^3k}{(2\pi)^3} \frac{i\omega_k}{2} \sum_{\lambda} \hat{A}^\dagger_{k,\lambda} \hat{E}_{k,\lambda} = \int \frac{d^3k}{(2\pi)^3} \frac{i\omega_k}{2} \hat{A}^\dagger_{-k,\lambda} \hat{E}_{-k,\lambda} = \int \frac{d^3k}{(2\pi)^3} \frac{i\omega_k}{2} \hat{A}_{+k,\lambda} \hat{E}^\dagger_{+k,\lambda}. \quad \text{(S.38)}$$

Consequently

$$\Delta \hat{H} = \int \frac{d^3k}{(2\pi)^3} \frac{i\omega_k}{2} \left( \hat{E}^\dagger_{k,\lambda} \hat{A}_{k,\lambda} - \hat{A}^\dagger_{k,\lambda} \hat{E}_{k,\lambda} \equiv [\hat{E}^\dagger_{k,\lambda}, \hat{A}_{k,\lambda}] \right)$$

$$= \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \frac{\omega_k}{2} (2\pi)^3 \delta(3)(\mathbf{k} - \mathbf{k} = \mathbf{0}) \quad \text{(S.39)}$$

$$\equiv E_{\text{vacuum}}$$

which is indeed a c-number constant, albeit badly divergent. \textbf{Q.E.D.}
Physically, the vacuum energy (S.39) is the net zero-point energy of all the oscillatory modes of the vector field theory. This energy is infinite for two reasons, one having do do with the infinite volume of space and the other with its perfect continuity. The infinite-volume divergence of $\int d^3x$ of a constant vacuum energy density manifest itself via the $(2\pi)^3\delta^{(3)}(0)$ factor, which is simply the Fourier transform of $\int d^3x(1)$. Indeed, had we quantized the theory in a very large but finite box, we would have obtained the $L^3$ volume factor in eq. (S.39) instead of the delta function. In other words, the vacuum has energy density

$$\frac{\text{Energy}}{\text{Volume}}|_{\text{Vacuum}} = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \frac{\omega_k}{2} = \int \frac{d^3k}{(2\pi)^3} \frac{3\omega_k}{2}.$$  \hspace{1cm} (S.40)

Alas, this integral diverges at large momenta so the vacuum energy density is also infinite. This is a generic problem of all Quantum Field Theories in a perfectly continuous space (and hence unlimitedly high momenta). Ultimately, this problem should be resolved by the fundamental theory of physics at ultra-short distances, whatever such theory might be.

Fortunately, for all practical purposes, we may safely disregard any c-number constant term in the Hamiltonian, even if such term is infinite — and that is exactly what we shall do in this class!

**Problem 2(e):**
Reversing eqs. (17), we have

$$\hat{A}_{k,\lambda} = \sqrt{\frac{C_{k,\lambda}}{2\omega_k}} \left( \hat{a}_{k,\lambda} - \hat{a}^+_{-k,\lambda} \right)$$  \hspace{1cm} (S.41)

and therefore, in the Schrödinger picture,

$$\hat{A}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx}}{2\omega_k} \sum_\lambda \sqrt{C_{k,\lambda}} e_\lambda(k) \left( \hat{a}_{k,\lambda} - \hat{a}^+_{-k,\lambda} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{e^{+ikx}}{2\omega_k} \sum_\lambda \sqrt{C_{k,\lambda}} e_\lambda(k) \hat{a}_{k,\lambda} - \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ikx}}{2\omega_k} \sum_\lambda \sqrt{C_{k,\lambda}} e_\lambda(-k) \hat{a}_{k,\lambda}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{e^{+ikx}}{2\omega_k} \sum_\lambda \sqrt{C_{k,\lambda}} \left( e^{+ikx} e_\lambda(k) \hat{a}_{k,\lambda} + e^{-ikx} e_\lambda^*(k) \hat{a}^+_{k,\lambda} \right).$$  \hspace{1cm} (S.42)

In the Heisenberg picture, the time-dependence of the quantum field $A(x, t)$ follows from
the time-dependence of the creation and annihilation operators,

\[ \hat{a}_{k,\lambda}^\dagger(t) = e^{+it\hat{H}}\hat{a}_{k,\lambda}^\dagger(0)e^{-it\hat{H}} \quad \text{and} \quad \hat{a}_{k,\lambda}(t) = e^{+it\hat{H}}\hat{a}_{k,\lambda}(0)e^{-it\hat{H}}. \]  

(S.43)

For the free Hamiltonian (18),

\[ [\hat{a}_{k,\lambda}, \hat{H}] = -\omega_k \times \hat{a}_{k,\lambda} \implies \hat{a}_{k,\lambda}(t) = e^{-it\omega_k} \hat{a}_{k,\lambda}(0), \]

\[ [\hat{a}_{k,\lambda}^\dagger, \hat{H}] = +\omega_k \times \hat{a}_{k,\lambda}^\dagger \implies \hat{a}_{k,\lambda}^\dagger(t) = e^{+it\omega_k} \hat{a}_{k,\lambda}^\dagger(0), \]  

(S.44)

similar to the scalar creation and annihilation operators we have studied in class. Consequently, substituting this time dependence into eq. (S.42), we arrive at

\[ \hat{A}(x, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \sum_\lambda \sqrt{C_{k,\lambda}} \left( e^{+ikx-\omega_k t}\varepsilon_\lambda(k) \hat{a}_{k,\lambda}(0) + e^{-ikx+\omega_k t}\varepsilon_\lambda^*(k) \hat{a}_{k,\lambda}^\dagger(0) \right) \]

\[ = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \sum_\lambda \sqrt{C_{k,\lambda}} \left( e^{-ik\cdot x}\varepsilon_\lambda(k) \hat{a}_{k,\lambda}(0) + e^{+ik\cdot x}\varepsilon_\lambda^*(k) \hat{a}_{k,\lambda}^\dagger(0) \right) \]

in accordance with eq. (19).

Problem 2(f):

Similar to part (e), reversing eqs. (17) gives us

\[ \hat{E}_{k,\lambda} = \frac{i}{2\sqrt{C_{k,\lambda}}} (\hat{a}_{k,\lambda} + \hat{a}_{k,\lambda}^\dagger), \]  

(S.46)

hence in the Schrödinger picture

\[ \hat{E}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx}}{2\omega_k} \sum_\lambda \frac{i\omega_k}{\sqrt{C_{k,\lambda}}} \varepsilon_\lambda(k) \left( \hat{a}_{k,\lambda} + \hat{a}_{k,\lambda}^\dagger \right) \]

\[ = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx}}{2\omega_k} \sum_\lambda \frac{i\omega_k}{\sqrt{C_{k,\lambda}}} \varepsilon_\lambda(k) \hat{a}_{k,\lambda} + \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ikx}}{2\omega_k} \sum_\lambda \frac{i\omega_k}{\sqrt{C_{k,\lambda}}} \varepsilon_\lambda(-k) \hat{a}_{k,\lambda} \]

\[ = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx}}{2\omega_k} \sum_\lambda \frac{i\omega_k}{\sqrt{C_{k,\lambda}}} \left( e^{ikx}\varepsilon_\lambda(k) \hat{a}_{k,\lambda} - e^{-ikx}\varepsilon_\lambda^*(k) \hat{a}_{k,\lambda}^\dagger \right). \]  

(S.47)
In the Heisenberg picture, this formula becomes

$$\hat{E}(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda} \frac{i\omega_{k,\lambda}}{\sqrt{C_{k,\lambda}}} \left( e^{-ik_{\mu}x^\mu} e_{\lambda}(k) \hat{a}_{k,\lambda}(0) - e^{+ik_{\mu}x^\mu} e_{\lambda}^*(k) \hat{a}_{k,\lambda}^\dagger(0) \right)_{k^0=+\omega_k}. \quad (S.48)$$

This takes care of the quantum electric field.

Now consider the scalar potential $\hat{A}^0$ — which is related to the electric field by the time-independent operatorial identity (10). Applying this identity to eq. (S.48), we obtain

$$\hat{A}^0(x, t) = \frac{-1}{m^2} \nabla \cdot \hat{E}(x, t) = \frac{-1}{m^2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda} \frac{i\omega_k}{\sqrt{C_{k,\lambda}}} \left( e^{-i\omega_k t} \times \left( \nabla e^{+ikx} \cdot e_{\lambda}(k) \right) \times \hat{a}_{k,\lambda} - e^{+i\omega_k t} \times \left( \nabla e^{-ikx} \cdot e_{\lambda}^*(k) \right) \times \hat{a}_{k,\lambda}^\dagger \right). \quad (S.49)$$

In this formula

$$\nabla e^{+ikx} \cdot e_{\lambda}(k) = e^{+ikx}(ik \cdot e_{\lambda}(k)) = e^{+ikx} \times i|k| \delta_{\lambda,0} \quad (S.50)$$

in the helicity basis, and likewise

$$\nabla e^{-ikx} \cdot e_{\lambda}^*(k) = e^{-ikx} \times (-i)|k| \delta_{\lambda,0}. \quad (S.51)$$

Thus, only the $\lambda = 0$ modes contribute to the scalar potential (S.49). Specifically,

$$\hat{A}^0(x, t) = \frac{-1}{m^2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \times \frac{i\omega_k}{\sqrt{C_{k,0}}} \times i|k| \left( e^{-i\omega_k t+ikx} \times \hat{a}_{k,0} + e^{+i\omega_k t-ikx} \times \hat{a}_{k,0}^\dagger \right) \quad (S.52)$$

where

$$\sqrt{C_{k,0}} = \frac{\omega_k}{m} \quad \Rightarrow \quad \frac{-1}{m^2} \times \frac{i\omega_{k,\lambda}}{\sqrt{C_{k,0}}} \times i|k| = +\frac{|k|}{m}. \quad (S.53)$$

Thus altogether,

$$\hat{A}^0(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \times \frac{|k|}{m} \times \left( e^{-ik_{\mu}x^\mu} \times \hat{a}_{k,0} + e^{+ik_{\mu}x^\mu} \times \hat{a}_{k,0}^\dagger \right)_{k^0=+\omega_k}. \quad (S.53).$$
Problem 2\(g\):

In light of similarity between eqs. (19) and (S.53), we may combine them into

\[
\hat{A}^\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\lambda \left( e^{-ikx} f^\mu(k,\lambda) \times \hat{a}_{k,0}(0) + e^{+ikx} f^{\mu*}(k,\lambda) \times \hat{a}_{k,0}^\dagger(0) \right)_{\nu^\mu=\omega_k} \tag{20}
\]

where

\[
f(k,\lambda) = \sqrt{C_{k,\lambda}} e_\lambda(k) \quad \text{and} \quad f^0(k,\lambda) = \frac{|k|}{m} \delta_{\lambda,0}. \tag{S.54}\]

Specifically,

\[
\text{for } \lambda = \pm 1, \quad f = e_\lambda(k), \quad f^0 = 0, \tag{S.55}\]

while

\[
\text{for } \lambda = 0, \quad f = \frac{\omega_k}{m} \frac{k}{|k|}, \quad f^0 = \frac{|k|}{m}. \tag{S.56}\]

In other words, the polarization 4–vectors \(f^\mu(k,\lambda)\) are precisely as in eq. (21).

It is easy to see that these 4–vectors obtain by Lorentz-boosting the purely spatial vectors \((0,e_\lambda(k))\) by the wave’s velocity \(v = k/\omega_k\). Indeed, for the transverse polarizations \(\lambda = \pm 1\), the vectors \(e_\lambda\) are perpendicular to the \(k\) and hence to the wave’s velocity \(v\), so they are left unchanged by the boost. On the other hand, for the longitudinal polarization \(\lambda = 0\), the vector \(e_\lambda\) is parallel to the velocity, so after the boost it becomes

\[
f^0(0,k) = \gamma v \cdot e_0(k) = \gamma \frac{|k|}{\omega_k} = \frac{|k|}{m}, \tag{S.57}\]

\[
f(0,k) = \gamma e_0(k) = \frac{\omega_k}{m} \frac{k}{|k|}.
\]

Finally, let’s verify the equations (22) for the polarization vectors:

\[
k_\mu f^\mu(k,\pm 1) = -k \cdot e_{\pm 1}(k) = 0, \tag{S.58}\]

\[
k_\mu f^\mu(k,0) = \omega_k f^0(k,0) - k \cdot f(k,0)
= \frac{\omega_k \times |k|}{m} - \frac{|k| \times \omega_k}{m} = 0, \tag{S.59}\]
\[ f^\mu(k, \pm 1) \times f^*_\mu(k, \mp 1) = -e_{\pm 1} \cdot e^*_{\mp 1} = -1, \quad (S.60) \]
\[ f^\mu(k, \pm 1) \times f^*_\mu(k, \mp 1) = -e_{\pm 1} \cdot e^*_{\mp 1} = 0, \quad (S.61) \]
\[ f^\mu(k, \pm 1) \times f^*_\mu(k, 0) = -(e_{\pm 1} \cdot e^*_0) \times \frac{\omega_k}{m} = 0 \times \frac{\omega_k}{m} = 0, \quad (S.62) \]
\[ f^\mu(k, 0) \times f^*_\mu(k, 0) = \left( \frac{|k|}{m} \right)^2 - \left( \frac{\omega_k}{m} \right)^2 = \frac{k^2 - \omega^2_k}{m^2} = -1. \quad (S.63) \]

**Problem 2(h):**
Each positive-frequency plane-wave mode \( e^{-ikx} f^\mu(k, \lambda) \) in eq. (20) satisfies
\[ (\partial^2 + m^2) \left( e^{-ikx} f^\mu(k, \lambda) \right) = 0 \quad \text{and} \quad \partial_\mu \left( e^{-ikx} f^\mu(k, \lambda) \right) = 0 \quad (S.64) \]
because \(-k^2 + m^2 = 0\) and \(k_\mu f^\mu(k, \lambda) = 0\). For the same reason, each negative-frequency mode \( e^{+ikx} f^*_\mu(k, \lambda) \) in eq. (20) also satisfies
\[ (\partial^2 + m^2) \left( e^{+ikx} f^*_\mu(k, \lambda) \right) = 0 \quad \text{and} \quad \partial_\mu \left( e^{+ikx} f^*_\mu(k, \lambda) \right) = 0. \quad (S.65) \]

From the space-time point of view, the quantum vector field (20) is a linear superposition of plane waves satisfying the classical field equations
\[ (\partial^2 + m^2) \hat{A}^\mu(x) = 0 \quad \text{and} \quad \partial_\mu \hat{A}^\mu(x) = 0. \quad (S.66) \]

By linearity of these equations, any linear superposition of solutions is a solution, so the free quantum field does obey the classical field equations.

**Problem 3(a):**
The simplest way to prove the lemma (23) is by direct inspection, component by component:
\[
\sum_\lambda f^i(k, \lambda)f^{*j}(k, \lambda) = \sum_{\lambda=\pm 1} e^i_\lambda(k)e^{*j}_\lambda(k) + \left( \frac{\omega^2_k}{m^2} = 1 + \frac{k^2}{m^2} \right) \times e^i_0(k)e^{*j}_0(k)
\]
\[ = \sum_{\lambda=\pm 1, 0} e^i_\lambda(k)e^{*j}_\lambda(k) + \frac{k^2}{m^2} \times \frac{k^i}{|k|} \times \frac{k^j}{|k|} = \delta^{ij} + \frac{k^ik^j}{m^2}, \quad (S.67) \]
\[
\sum_{\lambda} f_i(\mathbf{k}, \lambda) f_i^*(\mathbf{k}, \lambda) = f_i(\mathbf{k}, 0) f_i^*(\mathbf{k}, 0) = \frac{k^i \omega_k}{m^2}, \quad (S.68)
\]
\[
\sum_{\lambda} f^0(\mathbf{k}, \lambda) f^0(\mathbf{k}, \lambda) = \left| f^0(\mathbf{k}, 0) \right|^2 = \frac{k^2}{m^2} = -1 + \frac{\omega_k^2}{m^2}. \quad (S.69)
\]

Alternatively, we may use the fact that the three four-vectors \( f^\mu(\mathbf{k}, \lambda) \) (fixed \( \mathbf{k}, \lambda = -1, 0, +1 \)) are orthogonal to each other and also to the \( k^\mu = (\omega_k, \mathbf{k}) \). Furthermore, each \( (f(\mathbf{k}, \lambda))^2 = -1 \). Consequently, the symmetric matrix (in Lorentz indices \( \mu, \nu \)) on the left hand side of eq. (23) has to be (minus) the projection matrix onto four-vectors orthogonal to the \( k^\mu \), and that is precisely the matrix appearing on the right hand side of eq. (23) (note \( k^2 = m^2 \)).

**Problem 3(b):**
The operator product \( \hat{A}^\mu(x) \hat{A}^\nu(y) \) comprises \( \hat{a} \hat{a}, \hat{a}^\dagger \hat{a}^\dagger, \hat{a}^\dagger \hat{a} \) and \( \hat{a} \hat{a}^\dagger \) terms. The first three kinds of terms have zero matrix elements between vacuum states while \( \langle 0 | \hat{a}_{k, \lambda} \hat{a}_{k', \lambda'} | 0 \rangle = 2 \omega_k (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k'}) \delta_{\lambda, \lambda'} \). Consequently,

\[
\langle 0 | \hat{A}^\mu(x) \hat{A}^\nu(y) | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \sum_{\lambda} \left[ e^{-ik(x-y)} f^\mu(\mathbf{k}, \lambda) f^{*\nu}(\mathbf{k}, \lambda) \right]_{k^0 = +\omega_k}
\]

\[
= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[ \left( -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) e^{-ik(x-y)} \right]_{k^0 = +\omega_k}
\]

\[
= \left( -g^{\mu\nu} - \frac{1}{m^2} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[ e^{-ik(x-y)} \right]_{k^0 = +\omega_k}
\]

\[
\equiv \left( -g^{\mu\nu} - \frac{1}{m^2} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \right) D(x - y)
\]

in accordance with eq. (24).

**Problem 3(c):**
Consider a free scalar field \( \hat{\Phi} \) of the same mass \( m \) as the vector field. In class we saw that

\[
\langle 0 | \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle = D(x - y) \quad (S.71)
\]
so we may interpret eq. (24) as

$$\langle 0 | \hat{A}^\mu(x) \hat{A}^\nu(y) | 0 \rangle = \left( -g^{\mu\nu} - \frac{1}{m^2} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right) \langle 0 | \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle. \quad (S.72)$$

In class we have also learned that the scalar Feynman propagator

$$G_F^{\text{scalar}}(x - y) = \langle 0 | T \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle = \begin{cases} D(x - y) & \text{for } x^0 > y^0, \\ D(y - x) & \text{for } y^0 > x^0, \end{cases} \quad (S.73)$$

satisfies

$$\frac{\partial}{\partial x^0} G_F^{\text{scalar}}(x - y) = \langle 0 | T \partial_0 \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle \quad (S.74)$$

but

$$\frac{\partial^2}{(\partial x^0)^2} G_F^{\text{scalar}}(x - y) = \langle 0 | T \partial_0^2 \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle - i\delta^{(4)}(x - y). \quad (S.75)$$

Since the space derivatives \( \partial_\mu \) for \( \mu \neq 0 \) commute with the time-ordering, it follows that

$$\left( -g^{\mu\nu} - \frac{1}{m^2} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right) G_F^{\text{scalar}}(x - y) = \langle 0 | T \left( -g^{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m^2} \right) \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle + \frac{i}{m^2} \delta^{\mu_0} \delta^{\nu_0} \delta^{(4)}(x - y). \quad (S.76)$$

Now, for any differential operator \( \mathcal{D} \) acting on \( \hat{\Phi}(x) \) we have

$$\langle 0 | \mathcal{D} \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle = \mathcal{D}_x \langle 0 | \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle = \mathcal{D}_x D(x - y), \quad (S.77)$$

and consequently

$$\langle 0 | T \mathcal{D} \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle = \begin{cases} \langle 0 | \mathcal{D} \hat{\Phi}(x) \hat{\Phi}(y) | 0 \rangle & \text{for } x^0 > y^0, \\ \langle 0 | \hat{\Phi}(y) \mathcal{D} \hat{\Phi}(x) | 0 \rangle & \text{for } y^0 > x^0, \end{cases} \quad (S.78)$$

$$= \begin{cases} \mathcal{D}_x D(x - y) & \text{for } x^0 > y^0, \\ \mathcal{D}_x D(y - x) & \text{for } y^0 > x^0. \end{cases}$$
In particular, for
\[
\mathcal{D} = \left( -g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right)
\] (S.79)
we have
\[
\langle 0 \mid T \left( -g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) \hat{\Phi}(x) \hat{\Phi}(y) \mid 0 \rangle = \begin{cases} 
\left( -g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) D(x - y) & \text{for } x^0 > y^0, \\
\left( -g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) D(y - x) & \text{for } y^0 > x^0,
\end{cases}
\]
(\text{in light of eq. (S.72)})
\[
= \begin{cases} 
\langle 0 \mid \hat{A}^\mu(x) \hat{A}^{\nu}(y) \mid 0 \rangle & \text{for } x^0 > y^0, \\
\langle 0 \mid \hat{A}^{\nu}(y) \hat{A}^\mu(x) \mid 0 \rangle & \text{for } y^0 > x^0,
\end{cases}
\]
\[
\equiv \langle 0 \mid T \hat{A}^\mu(x) \hat{A}^{\nu}(y) \mid 0 \rangle
\] (S.80)
for the un-modified time ordering T on the last line.

Finally, let’s plug eq. (S.80) into the RHS of eq. (S.76). Thus way, we obtain
\[
\left( -g^{\mu\nu} - \frac{1}{m^2} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right) G^{\text{scalar}}_F(x - y) = \langle 0 \mid T \hat{A}^\mu(x) \hat{A}^{\nu}(y) \mid 0 \rangle + \frac{i}{m^2} \delta^{\mu 0} \delta^{\nu 0} \delta^{(4)}(x - y)
\]
\[
\equiv \langle 0 \mid T^* \hat{A}^\mu(x) \hat{A}^{\nu}(y) \mid 0 \rangle
\]
(\text{for the modified time-ordering})
\[
\overset{\text{def}}{=} G_F^{\mu\nu}(x - y)
\] (S.81)
— the Feynman propagator for the vector field. This proves the first line of eq. (25).

To prove the second line, we use the momentum-space form of the scalar propagator,
\[
G^{\text{scalar}}_F(x - y) = \int \frac{d^4 k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i0}.
\] (S.82)
Consequently,
\[ G_{\mu\nu}^F (x - y) = \left( -g^{\mu\nu} - \frac{1}{m^2} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \right) G_{\text{scalar}}^F (x - y) \]
\[ = \left( -g^{\mu\nu} - \frac{1}{m^2} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \right) \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i0} \]  \hspace{1cm} (S.83)

\[ = \int \frac{d^4 k}{(2\pi)^4} \left( -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) \times \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i0} \].

Q.E.D.

Problem 3(d):

The free massive vector field has classical Lagrangian
\[ \mathcal{L} = -\frac{1}{4} F_{\nu\mu} F^{\nu\mu} + \frac{1}{2} m^2 A_\mu A^\mu \]
\[ = -\frac{1}{2} (\partial_\nu A_\mu) (\partial^\nu A^\mu - \partial^\mu A^\nu) + \frac{1}{2} m^2 A_\mu A^\mu \]  \hspace{1cm} (S.84)

so the action may be written as
\[ S \equiv \int d^4 x \, \mathcal{L} = \frac{1}{2} \int d^4 x \, A_\mu(x) \mathcal{D}^{\mu\nu} A_\nu(x) \]  \hspace{1cm} (S.85)

where
\[ \mathcal{D}^{\mu\nu} = (\partial^2 + m^2) \times g^{\mu\nu} - \partial^\mu \partial^\nu \]  \hspace{1cm} (S.86)

is a second-order differential operator. Acting with this operator on the Feynman propagator (25), we have
\[ \mathcal{D}^{\mu\nu} G_{\nu\lambda}^F (x - y) = \left( (\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right) \left( -g_{\nu\lambda} - \frac{1}{m^2} \partial_{\nu} \partial_{\lambda} \right) G_{\text{scalar}}^F (x - y) \]
\[ = \left( -\delta^\mu_\lambda \times (\partial^2 + m^2) + \partial^\mu \partial_\lambda \right) \times \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \]  \hspace{1cm} (S.87)

\[ = \left( -\delta^\mu_\lambda \times (\partial^2 + m^2) + 0 \times \partial^\mu \partial_\lambda \right) \times G_{\text{scalar}}^F (x - y) \]
\[ = -\delta^\mu_\lambda \times (\partial^2 + m^2) G_{\text{scalar}}^F (x - y) = -i \delta^{(4)}(x - y) \]
\[ = +i \delta^\mu_\lambda \delta^{(4)}(x - y) \]

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in accordance with eq. (28). This means that the Feynman propagator of the massive vector fields is indeed a Green’s function of the differential operator (S.86).